1 Set Theory

Syllabus

Basic Concepts of Set Theory : Definitions, Inclusion, Equality of Sets, Cartesian product, The Power Set, Some operations on Sets, Venn Diagrams, Some Basic Set Identities.

Contents

1.1 Sets

A well defined collection of objects is called a set.

Remarks

If objects are not well defined then the collection cannot be a set.

e.g. 1-chair, 1-book, 1-table, 1-pen, 1-purse etc. This collection of objects cannot form a set.

But the collection of objects where objects are books, the collection will be known as set.

Examples

1. Library is a set of books.
2. Classroom is a set of chairs.
3. Class is a set of students.
4. Family is a set of persons.
5. Set of all straight lines in a 2-D plane.
6. Set of all triangles in a 2-D plane.

Some special sets.

N is a set of natural numbers.

$$N = \{1, 2, 3 ----\}$$

I or Z is a set of integers.

$$I = \{----, -2, -1, 0, 1, 2, 3, ----\}$$

W or Z^+ or I^+ is a set of positive integers.

$$W = \{0, 1, 2, 3, ----\}$$

Q is set of rational numbers.

$$Q = \{x; x = \frac{p}{q}, p, q \in I \text{ and } q \neq 0\}$$

$\Re$ is a set of real numbers

$$\Re = \{x; x = a + b\sqrt{p}, \text{ where } a, b \in Q \text{ and } p \in Q_+\}$$

$\mathbb{C}$ is the set of complex numbers.

$$\mathbb{C} = \{x; x = a + ib; a, b \in \Re \text{ and } i = \sqrt{-1}\}$$

Object belonging to a set is called an element of the set. We generally use capital letters A, B, C..... to denote sets and lower case letters a, b, c, ... to denote elements of a set.

The symbol '$\in$' is used to indicate element is in the set and is known as 'belongs to' or 'is an element of'. Hence x is an element of X is written as

$$x \in X$$

The symbol '$\notin$' is used to indicate element is not in the set and is known as 'does not belong to' or 'is not an element of'. Hence y is not an element of X is written as

$$y \notin X$$

1.2 Methods of Discribing Sets

The following are the most common methods of discribing sets.

1.2.1 Roster Method

A set may be discribed by listing all its elements.

e.g. $N = \{1, 2, 3, —\}$

Set of natural numbers.

This method is called roster method or **listing** method.

1.2.2 Set Selector Method (or Set Builder Form or Property Method)

Sometimes a set is discribed by means of some property which characterizes all the elements of the set (that is, the property is shared by all the elements of the set and if a certain object has the property, then it belongs to the set). A set S characterized by a property p may be written as

$$S = \{x : p(x)\}$$

Example

1. $A = \{x \,;\, x^2 - 4 = 0, x \in \Re\}$

A is set of those elements which satisfies equation $x^2 - 4 = 0$ and x is a real number.

$$x = 2 \text{ and } x = -2$$

$$A = \{-2, 2\}$$

2. $B = \{x \,;\, 4 \leq x \leq 11, x \in N\}$

i.e. B is a set of natural numbers such that x lies between natural numbers 4 and 11.

$$B = \{4, 5, 6, 7, 8, 9, 10, 11\}$$

3. $C = \{x \,;\, x^3 + 1 = 0, x \in \Re\}$

i.e. C is a set of real numbers x such that x satisfies equation $x^3 + 1 = 0$. Only $x = -1$ real number satisfies the equation $x^3 + 1 = 0$.

$\therefore \quad C = \{-1\}$

1.3 Subsets

Let A and B be two sets. If each element of set A is an element of set B then 'A is called subset of B' and is denoted by,

$$'A \subset B'$$

Sometime it is said to be 'A is contained in B'.

We also say that 'B contains A', or 'B is superset of A', and is expressed as '$B \supset A$'.

Hence '$A \subset B$' or '$B \supset A$' are the same facts in two different expressions.

If 'A is a subset of B' i.e. '$A \subset B$' then 'B is a superset of A', i.e. '$B \supset A$'.

If 'A is not subset of B', we write

$$A \not\subset B$$

1.3.1 Proper Subset

If 'A is a subset of B' and number of elements in A and B are not same i.e. $A \neq B$ then A is known as proper subset of B.

Examples

1. $A = \{a, b, c, d\}$

$B = \{x, y, z\}$

$C = \{a, b, c, d, x, y, z\}$

$D = \{a, b, 1, 2\}$

$\Rightarrow \quad A \subset C$

$B \subset C$

$D \not\subset C$

$D \not\subset A$ etc.

2. $N \subset W$

$W \subset I$

$I \subset Q$

$$\Re \subset \mathbb{C}$$

$$\Rightarrow N \subset W \subset I \subset Q \subset \Re \subset \mathbb{C}$$

1.4 Equality of Sets

Two sets A and B are equal sets if 'A ⊂ B' and 'B ⊂ A' i.e. 'A is a subset of B' and also 'B is a subset of A'.

Hence $A = B \Rightarrow A \subset B$ and $B \subset A$

or $A = B \Rightarrow A \supset B$ and $B \supset A$

1.5 The Empty Set or Null Set or Void Set

The set having no element is called empty set or null set or void set.

Empty set is denoted by ∅ or { }.

Examples

1. $A = \{ x ; x^2 + 4 = 0 \text{ and } x \in \Re \}$

   $A = \varnothing$

   as no real number satisfies equation $x^2 + 4 = 0$.

2. $B = \{ y; y < 0 \text{ and } y \in N \}$

   $B = \varnothing$ as no natural number is less than zero.

Results

1. The empty set is subset of every set i.e. $\varnothing \subset A$.
2. Every set is subset of itself i.e. $A \subset A$.

1.6 Universal Set

In any application of set theory, all the sets under consideration will likely be subsets of a fixed set. We call this set the universal set and denote it by capital letter X.

Thus a non-empty set of which all the sets under consideration are subsets is called universal set.

1.7 Ordered Pair

Let A and B be two sets. Let $a \in A$ and $b \in B$. Then (a, b) denotes **ordered pair** and a is known as first co-ordinate of the ordered pair (a, b) and b as second co-ordinate of the ordered pair (a, b).

1.7.1 Cartesian Product

Let A and B be two sets then $A \times B$ is called Cartesian product of A and B.

$A \times B$ is a set of all distinct ordered pairs, in which first co-ordinate of ordered pair is from set A and second co-ordinate is from set B.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Let $\quad A = \{1, 2, 3\}, B = \{a, b\}$

then $\quad A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$

$$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$(1, 2) \notin A \times B$

as $\quad 2 \notin B$

$(a, b) \notin A \times B$

as $\quad a \notin A$

$(b, 3) \notin A \times B$

as $\quad b \notin A$ and $3 \notin B$

$\Rightarrow \quad (a, b) \notin A \times B$, either $a \notin A$ or $b \notin B$ or both.

Hence $\quad A \times B \neq B \times A$

So, Cartesian product is non commutative, if A and B are non empty and diffferent sets. i.e. $A \neq \varnothing$, $B \neq \varnothing$ and $A \neq B$.

Similarly $A \times B \times C$ is a set of all distinct ordered triple.

$$A \times B \times C = \{(a, b, c) : a \in A \text{ and } b \in B \text{ and } c \in C\}$$

this way,

$$A_1 \times A_2 \times A_3 \ldots \times A_n = \{(a_1, a_2, a_3, \ldots a_n) : a_1 \in A_1, a_2 \in A_2, \ldots a_i \in A_i, \ldots a_n \in A_n\}$$

is set of ordered n-tuples.

Remarks :

1. If A contains 3 elements and B contains 4 elements then $A \times B$ contains $3 \times 4 = 12$ elements.

   i.e. $|A \times B| = m \cdot n$

   if $|A| = m$ and $|B| = n$

2. $A \times B \times C = \{(a, b, c) ; a \in A, b \in B \text{ and } c \in C\}$

   $A \times (B \times C) = \{(a, (b, c)) ; a \in A, (b, c) \in B \times C\}$

$$(A \times B) \times C = \{((a, b), c) ; (a, b) \in A \times B, c \in C\}$$

All these $A \times B \times C$, $(A \times B) \times C$, $A \times (B \times C)$ are different.

Normally product of A, B, C, we define as $A \times B \times C$.

Theorem :

If A, B, C are sets then,

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

iv) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof :

Let $(a, b) \in A \times (B \cap C)$

$\Leftrightarrow a \in A$ and $b \in (B \cap C)$

$\Leftrightarrow a \in A$ and ($b \in B$ and $b \in C$)

$\Leftrightarrow$ ($a \in A$ and $b \in B$) and ($a \in A$ and $b \in C$)

$\Leftrightarrow (a, b) \in (A \times B)$ and $(a, b) \in (A \times C)$

$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$

$\Rightarrow A \times (B \cap C) = (A \times B) \cap (A \times C)$

Students are advised to prove the other three results as an exercise.

Remarks :

1. If $\Re$ is the set of real number then $\Re \times \Re$ we represent by $\Re^2$ and $\Re \times \Re \times \Re$ by $\Re^3$.
2. $\Re \times \Re = \Re^2$ is called the Euclidean plane.

1.8 Class of Sets or Family of Sets

If the elements of a set are sets themselves, then such a set is called as class of sets or family of sets or set of sets.

1.9 Power Set

It S is any set, then set of all subsets of S is called power set of S and is denoted by P(S).

Example

$$S = \{1, 2, 3\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

So if S contains 3 elements power set of S, i.e. P(S) will contain 2^3, i.e. 8 elements.

Remark

If S has n elements then S will have 2^n subsets and set of these 2^n subsets will be known as power set.

Theorem

If a finite set S has n elements, then power set of S has 2^n elements.

Proof

Let us form all the subsets of S. $\emptyset$ is subset of every set. Taking 1 element at a time ${}^nC_1$ subsets can be formed out of S. Similarly ${}^nC_2$ subsets will be formed taking 2 elements of S at a time. This way total number of subsets will be

$$1 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_r + \dots + {}^nC_n = (1 + 1)^n \quad \text{... using binomial theorem.}$$

$$= 2^n$$

Hence total number of subsets will be 2^n.

Example

$$S = \{1, 2\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

P(s) is power set of S.

1.9.1 Power Set of Power Set of S

$$\begin{aligned} P(P(S)) = \{&\emptyset, \{\emptyset\}, \{\{1\}\}, \{\{2\}\}, \{\{1,2\}\}, \{\emptyset, \{1\}\}, \\ &\{\emptyset, \{2\}\}, \{\emptyset, \{1, 2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1, 2\}\}, \\ &\{\{2\}, \{1, 2\}\}, \{\emptyset, \{1\}, \{2\}\}, \{\emptyset, \{1\}, \{1, 2\}\}, \\ &\{\{1\}, \{2\}, \{1, 2\}\}, \{\emptyset, \{2\}, \{1, 2\}\}, \\ &\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}\} \end{aligned}$$

S contain 2 elements.

P(S) contain 2^2, i.e. 4 elements.

Then P(P(S)) will contain 2^4 i.e. 16 elements.

So P(P(P(S))) will contain 2^{16} elements etc.

Examples

$$S = \{1, 2\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

1. $1 \in S$ is true.
2. $1 \subset S$ is false. As 1 is an element of S not a subset of S.
3. $\{1\} \subset S$ is true. As {1} is a subset of S.
4. $\{1\} \in S$ is false. As {1} is not an element of S.
5. $\{1\} \in P(S)$ is true.
6. $\emptyset \subset P(S)$ is true.
7. $\emptyset \in P(S)$ is true

Statement 6 and 7 both are correct as $\emptyset$ is an element of P(S) also $\emptyset$ is subset of every set.

8. $\{1\} \in P(S)$ is true.
9. $\{\{1\}\} \in P(S)$ is false.
10. $\{\{1\}\} \in P(P(S))$ is true.
11. $\{\{1\}\} \subset P(P(S))$ is false.

Remark

When a is an element of S then it in denoted by $a \in S$. When A is a subset of S it is denoted by $A \subset S$. Hence for connection between element and set we use '$\in$' and for connection between two sets we use '$\subset$' symbol.

Examples

1. Determine whether each of the following statements is true or false.

a) $\emptyset \subseteq \emptyset$

True. As $\emptyset$ is subset of every set.

b) $\emptyset \in \emptyset$

False. As $\emptyset$ is not an element of $\emptyset$ set.

c) $\emptyset \subseteq \{\emptyset\}$

True. As $\emptyset$ is subset of each set and $\{\emptyset\}$ is a singleton set containing $\emptyset$ as element.

d) $\varnothing \in \{\varnothing\}$

True. Reason same as (c).

e) $\{\varnothing\} \subseteq \{\varnothing\}$

True. Every set is subset of itself.

f) $\{a, b\} \subseteq \{a, b, c, \{a, b, c\}\}$

True : Let $\{a, b\} = A$

$\{a, b, c, \{a, b, c\}\} = B$

then $\forall x \in A \Rightarrow x \in B \therefore A \subset B$

g) $\{a, b\} \in \{a, b, c, \{a, b, c\}\}$

False. As $\{a, b\}$ in not an element of $\{a, b, c, \{a, b, c\}\}$

h) $\{a, b, c\} \in \{a, b, c, \{a, b, c\}\}$

True. As $\{a, b, c\}$ is an element of $\{a, b, c, \{a, b, c\}\}$

i) $\{a, b, c\} \subset \{a, b, c, \{a, b, c\}\}$

True.

Let $A = \{a, b, c\}$

$B = \{a, b, c, \{a, b, c\}\}$

Then $\forall x \in A \Rightarrow x \in B$ hence $A \subset B$.

j) $\{a, \varnothing\} \in \{a, \{a, \varnothing\}\}$

True.

1.10 Venn-Euler Diagrams or Venn Diagrams

We often use pictures in mathematics. The relation between sets can be conveniently illustrated by certain diagrams called **Venn diagrams.**

In Venn diagrams a universal set X is represented by a large reactangle and subsets of X by regions enclosed within a closed curve generally circles, lying in the region enclosed by the rectangle.

If a sets B is a subset of A, the circle representing B is drawn inside the circle representing A. If A and B are disjoint sets , then the circles representing A and B are drawn in such a way that they have no common area.

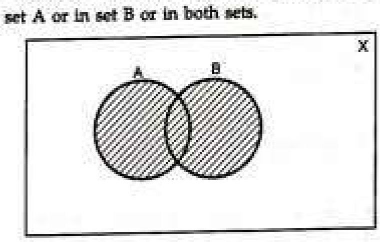
If A and B are not disjoint, then the circles representing A and B are drawn in such a way that they have some common area.

1.11 Operations on Sets

Sets can be combined in many different ways so as to produce new sets.

1.11.1 Union of Two Sets

Let A and B be two sets. Then the union of A and B is set of all those elements which are either in set A or in set B or in both sets.



A∪B

Union of sets A and B is denoted by $A \cup B$.

Which is read as 'A union B'.

Symbolically,

$A \cup B = \{x \,;\, x \in A \text{ or } x \in B\}$

Remark

In mathematics, when we use 'or', i.e. $x \in A$ or $x \in B$, we do not exclude the possibility that x is an element of both A and B.

Examples

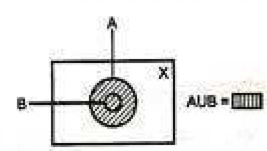
1. $A = \{1, 2, 3\}, B = \{x, y, z\}$

   $A \cup B = \{1, 2, 3, x, y, z\}$
2. $A = \{a, b, c, d\}, B = \{a, b, 1, 2\}$

   $A \cup B = \{a, b, c, d, 1, 2\}$
3. $A = \{0, 1, 2, 3, 4\}, B = \{2, 4, 6, 8\}$

   $A \cup B = \{0, 1, 2, 3, 4, 6, 8\}$
4. $A = \{1, 2, 3, 4, \}, B = \{1, 3\}, B \subseteq A$

   $A \cup B = \{1, 2, 3, 4\} = A$



Remark :

If $B \subset A$, then $A \cup B = A$

1.11.2 Intersection of Two Sets

Let A and B be two sets. The intersection of A and B is the set of all elements which are in A and also in B. i.e. set of common elements of A and B is intersection of sets A and B. We denote the intersection of A and B by $A \cap B$

$$A \cap B = \{x \,;\, x \in A \text{ and } x \in B\}$$

Examples

1. $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

   $A \cap B = \varnothing$

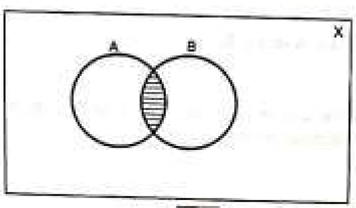
   intersection is null set as no element is common in A and B.

2. $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 6, 7\}$

   $A \cap B = \{1, 2\}$

   as $1 \in A$ and $1 \in B$ also $2 \in A$ and $2 \in B$.

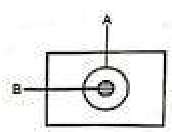
3. $A = \{1, 2, 3, 4\}$, $B = \{1, 3\}$, $B \subset A$ then $A \cap B = \{1, 3\} = B$



$A \cap B$

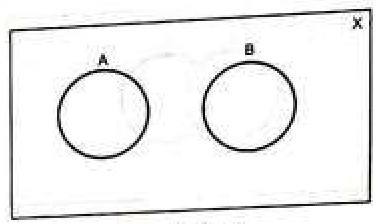
Remark :

If $B \subset A$, then $A \cap B = B$



1.11.3 Disjoint Sets

Set A and set B are called disjoint sets if no element is common to A and B. i.e. A and B are disjoint sets then $A \cap B = \varnothing$.



$A \cap B = \varnothing$

Examples

1. $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

   $A \cap B = \varnothing$

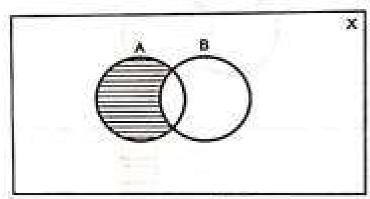
2. $A = \{x, y, z\}$, $B = \{a, b, c\}$

   $A \cap B = \varnothing$

1.11.4 Difference of Two Sets

Let A and B be two sets. The difference of A and B is denoted by 'A – B' or 'A – B' is set of all those elements of A which are not in B.

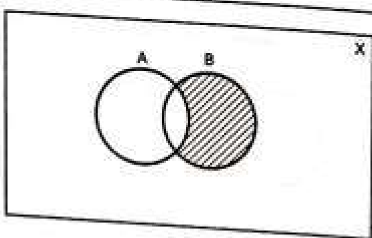
$$A - B = \{x \,;\, x \in A \text{ and } x \notin B\}$$



$A - B =$

Similarly

$$B - A = \{y ; y \in B \text{ and } y \notin A\}$$



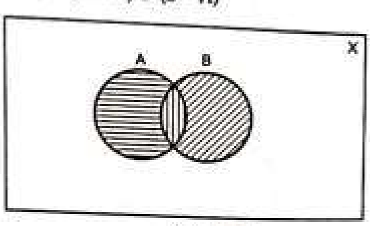
B – A =

Example

$$A = \{1, 2, 3, 4\},\ B = \{4, 5, 6, 7\}$$
$$A - B = \{1, 2, 3\}$$
$$B - A = \{5, 6, 7\}$$
$$A \cap B = \{4\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$$

$\Rightarrow$ A – B, B – A and $A \cap B$ are pairwise disjoint sets and

$$A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$$



A – B
B – A
A ∩ B

1.11.5 Symmetric Difference of Two Sets

Let A and B be two sets then symmetric difference of two sets A and B is denoted by $A \Delta B$ or $A \oplus B$ [read as A ring sum B].

And

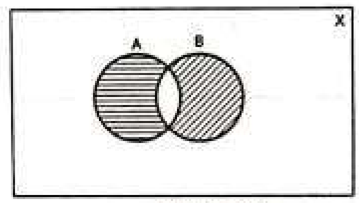
$$A \oplus B = (A - B) \cup (B - A)$$

Hence, if $x \in A \oplus B$

$\Rightarrow$ x is an element of exactly one of A and B or $x \in$ either A

or $x \in$ either A or $x \in B$ but x does not belong to both.

i.e. $A \oplus B = (A \cup B) - (A \cap B)$



A ⊕ B =

1.11.6 Complement of a Set

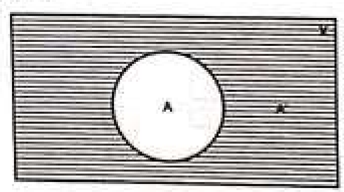
Let A be any set, then the complement of set A is denoted by A′ or $\bar{A}$ or A^C. Complement of A is set of those elements which are in universal set X but are not in set A.

$\therefore$ $A' = X - A$

$A' = \{x; x \in X \text{ and } x \notin A\}$

It is obvious that the element belong to universal set. Hence it is enough to say that

$A' = \{x ; x \notin A\}$

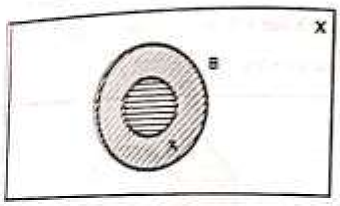


A′ =

Example 1.1 *Let A and B be sets such that $(A \cup B) \subseteq B$ and $B \not\subset A$. Draw the corresponding Venn diagram.*

Solution :

Let $A \subset B$



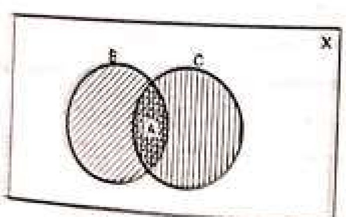
$A \cup B$ = [hatched] = B

then $A \cup B = B$

and every set is subset of itself hence $A \cup B \subseteq B$ also.

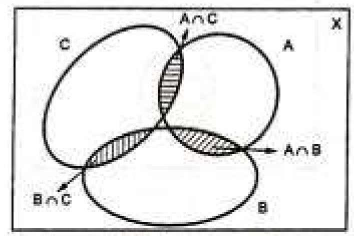
Example 1.2 *Let A, B and C be sets such that $A \subseteq B$, $A \subseteq C$, $(B \cap C) \subseteq A$ and $A \subseteq B \cap C$. Draw the corresponding Venn diagram.*

Solution :



Example 1.3 *Let A, B and C be sets such that $(A \cap B \cap C) = \emptyset$, $(A \cap B) \neq \emptyset$, $(A \cap C) \neq \emptyset$, $(B \cap C) \neq \emptyset$. Draw the corresponding Venn diagram.*

Solution :

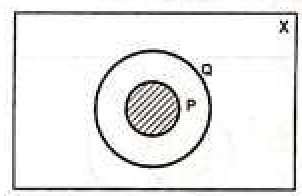


$A \cap B$ [hatched], $A \cap C$ [horizontal lines]

$B \cap C$ [vertical lines] and $A \cap B \cap C = \emptyset$

Example 1.4 *What can you say about the sets P and Q if a) $P \cap Q = P$? b) $P \cup Q = P$?*

Solution : a) $P \cap Q = P$

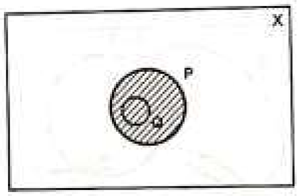


$P \cap Q$ [hatched]

From Venn diagram it is clear that

$$P \cap Q = P \quad \Rightarrow \quad P \subset Q$$

b) $P \cup Q = P$



$P \cup Q =$

$$P \cup Q = P \Rightarrow Q \subset P$$

Example 1.5 *Let $A = \{a, b, \{a, c\}, \varnothing\}$, determine the following sets.*

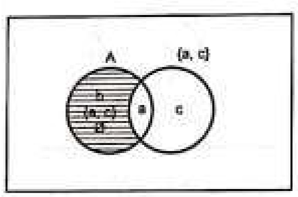
a) $A - \{a\}$ b) $A - \varnothing$ c) $A - \{a, c\}$

Solution : a) $A - \{a\} : \{b, \{a, c\}, \varnothing\}$

b) $A - \varnothing : \{a, b, \{a, c\}\}$

c) $A - \{a, c\} : \{b, \{a, c\}, \varnothing\}$

as $\{a, c\}$ is a set containing two elements a and c.



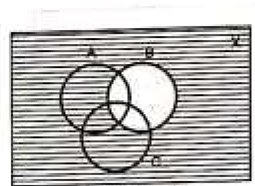
$A - \{a, c\} =$

Example 1.6 *Using Venn diagram show that :*

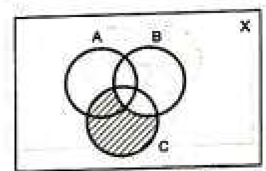
$$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$$

Solution :

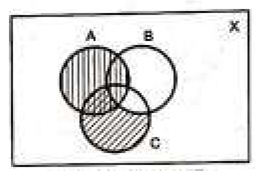
$$\overline{B} = \{x ; x \notin B\}$$



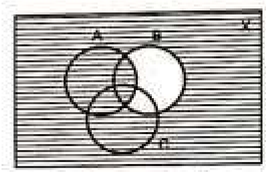
① $\overline{B} =$



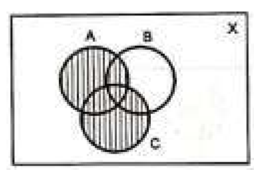
② $\overline{B} \cap C =$



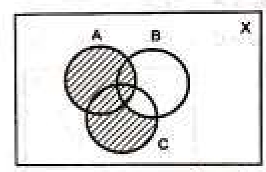
③ $A \cup (\overline{B} \cap C) =$ and



④ $A \cup \overline{B} =$



⑤ $A \cup C =$



⑥ $(A \cup \overline{B}) \cap (A \cup C) =$

From ③ and ⑥,

$$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$$

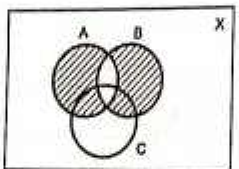
Example 1.7 *Using Venn diagram, prove or disprove.*

i) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

ii) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$

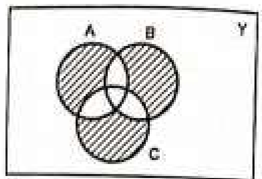
Solution : i) $A \oplus B = (A \cup B) - (A \cap B)$

i.e. elements which are either in A or in B but not in both A and B.



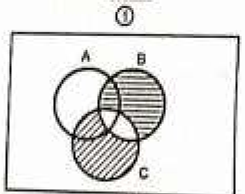
$A \oplus B =$ ▨

①



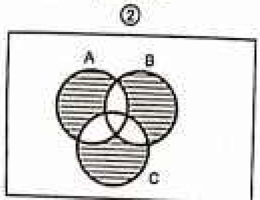
$(A \oplus B) \oplus C$

②



$B \oplus C =$ ▨

③ and ▤



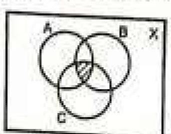
$A \oplus (B \oplus C)$

④

From ② and ④,

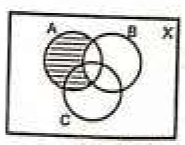
$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

II) $A \cap B \cap C = A - [(A - B) \cup (A - C)]$



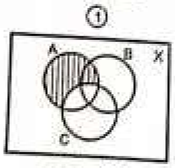
$A \cap B \cap C =$ ▨

①



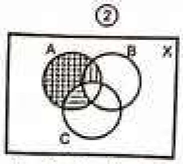
$A - B =$ ▤

②



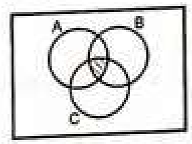
$A - C =$ ▥

③



$(A - B) \cup (A - C) =$ ▦ ▤ and ▥

④



$A - [(A-B) \cup (A-C)] =$ ▧

⑤

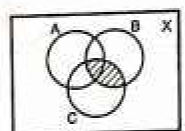
From ① and ⑤

$$A \cap B \cap C = A - [(A - B) \cup (A - C)]$$

Example 1.6 *Using Venn diagrams show that*

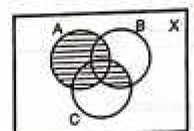
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution :



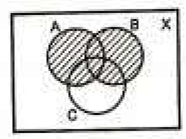
$B \cap C =$ ▨

①



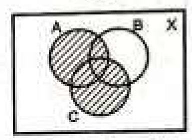
$A \cup (B \cap C) =$ ▤

②



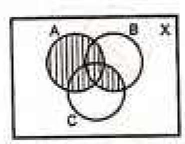
$A \cup B =$ ▨

③



$A \cup C =$ ▨

④



$(A \cup B) \cap (A \cup C) =$ ▥

⑤

From ① and ⑤,

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Example 1.9 *Let A, B, C be sets. Under what conditions the following statements are true ?*

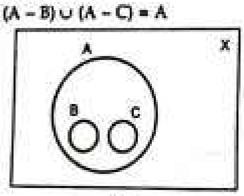
i) $(A - B) \cup (A - C) = A$

ii) $(A - B) \cup (A - C) = \emptyset$

Solution : i) $(A - B) \cup (A - C) = A$

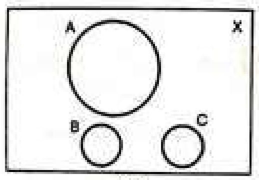
If $B \subset A$ and $C \subset A$

then $(A - B) \cup (A - C) = A$



①

OR



②

Or A, B, C are disjoint sets.

i.e. $A \cap B \cap C = \emptyset$

Then $(A - B) \cup (A - C) = A$

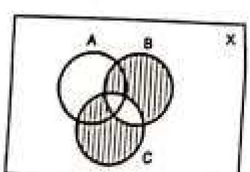
ii) $(A - B) \cup (A - C) = \emptyset$ is true.

If A is empty set i.e. $A = \emptyset$

Example 1.10 *Prove the following using Venn diagram.*

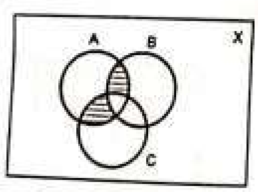
$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

Solution :



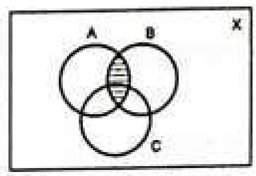
$B \oplus C =$

①



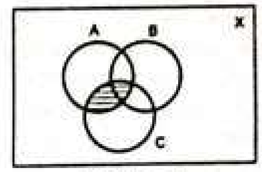
$A \cap (B \oplus C) =$

②



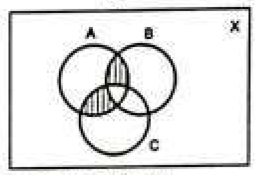
$A \cap B =$

③



$A \cap C =$

④



$(A \cap B) \oplus (A \cap C) =$

⑤

From ① and ⑤

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

1.12 Some Basic Set Identities

Theorem :

Let A, B, C be any subsets of a set X, then

1. Idempotent Laws

i) $A \cup A = A$

ii) $A \cap A = A$

2. Commutative Laws

i) $A \cup B = B \cup A$

ii) $A \cap B = B \cap A$

3. Associative Laws

i) $A \cup (B \cup C) = (A \cup B) \cup C$

ii) $A \cap (B \cap C) = (A \cap B) \cap C$

4. Distributive Laws

'$\cap$' distributes over '$\cup$' from the left and also from the right.

i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

'$\cup$' Distributes over '$\cap$' from the left and also from the right.

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

5. DeMorgan's Laws

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

6. Absorption Laws

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

7. Double complement

$(A')' = A$

Proof :

1) i) $A \cup A = A$

Let $x \in A \cup A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow x \in A$

$\Rightarrow A \cup A \subset A$ ... (1)

Now let $y \in A \Rightarrow y \in A$ or $y \in A$

$\Rightarrow y \in A \cup A$

$\Rightarrow A \subset A \cup A$ ... (2)

From (1) and (2),

$$\boxed{A \cup A = A}$$

ii) $A \cap A = A$

Let $x \in A \cap A \Rightarrow x \in A$ and $x \in A$

$\Rightarrow x \in A$

$\Rightarrow A \cap A \subset A$ ... (1)

Now let $y \in A \Rightarrow y \in A$ and $y \in A$

$\Rightarrow y \in A \cap A$

$\Rightarrow A \subset A \cap A$ ... (2)

From (1) and (2),

$$\boxed{A \cap A = A}$$

2) i) $A \cup B = B \cup A$

Let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$\Rightarrow x \in B \cup A$

$\Rightarrow A \cup B \subset B \cup A$ ... (1)

Now let $y \in B \cup A \Rightarrow y \in B$ or $y \in A$

$\Rightarrow y \in A$ or $y \in B$

$\Rightarrow y \in A \cup B$

$\Rightarrow B \cup A \subset A \cup B$ ... (2)

From (1) and (2),

$$\boxed{A \cup B = B \cup A}$$

ii) $A \cap B = B \cap A$

Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in B \cap A$

$\Rightarrow A \cap B \subset B \cap A$ ... (1)

Now let $y \in B \cap A \Rightarrow y \in B$ and $y \in A$

$\Rightarrow y \in A$ and $y \in B$

$\Rightarrow y \in A \cap B$

$\Rightarrow B \cap A \subset A \cap B$ ... (2)

From (1) and (2),

$$\boxed{A \cap B = B \cap A}$$

3) i) $A \cup (B \cup C) = (A \cup B) \cup C$

Let $x \in A \cup (B \cup C)$

$\Rightarrow x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A$ or ($x \in B$ or $x \in C$)

$\Rightarrow$ ($x \in A$ or $x \in B$) or $x \in C$

$\Rightarrow$ ($x \in A \cup B$) or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$

$\Rightarrow A \cup (B \cup C) \subset (A \cup B) \cup C$ ... (1)

Now let $y \in (A \cup B) \cup C$

$\Rightarrow y \in (A \cup B)$ or $y \in C$

$\Rightarrow$ ($y \in A$ or $y \in B$) or $y \in C$

$\Rightarrow y \in A$ or ($y \in B$ or $y \in C$)

$\Rightarrow y \in A$ or $y \in (B \cup C)$

$\Rightarrow y \in A \cup (B \cup C)$

$\Rightarrow (A \cup B) \cup C \subset A \cup (B \cup C)$ ... (2)

From (1) and (2),

$$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$$

ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Let $x \in (A \cap B) \cap C$

$\Rightarrow x \in (A \cap B)$ and $x \in C$

$\Rightarrow$ ($x \in A$ and $x \in B$) and $x \in C$

$\Rightarrow x \in A$ and ($x \in B$ and $x \in C$)

$\Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in (A \cap (B \cap C))$

$\Rightarrow (A \cap B) \cap C = A \cap (B \cap C)$ ... (1)

Now let $y \in A \cap (B \cap C)$

$\Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow y \in A$ and ($y \in B$ and $y \in C$)

$\Rightarrow$ ($y \in A$ and $y \in B$) and $y \in C$

$\Rightarrow y \in (A \cap B)$ and $y \in C$

$\Rightarrow y \in (A \cap B) \cap C$

$\Rightarrow A \cap (B \cap C) \subset (A \cap B) \cap C$ ... (2)

From (1) and (2),

$$\boxed{(A \cap B) \cap C = A \cap (B \cap C)}$$

4) i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and $x \in (B \cup C)$

$\Rightarrow x \in A$ and [$x \in B$ or $x \in C$]

$\Rightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Rightarrow$ ($x \in A \cap B$) or $x \in A \cap C$)

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ ... (1)

Net let, $y \in (A \cap B) \cup (A \cap C)$

$\Rightarrow y \in (A \cap B)$ or $y \in (A \cap C)$

$\Rightarrow$ ($y \in A$ and $y \in B$) or ($y \in A$ and $y \in C$)

$\Rightarrow y \in A$ or ($y \in B$ and $y \in C$)

$\Rightarrow y \in A$ or $y \in (B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cap B) \cup (A \cap C) \subset A \cup (B \cap C)$ ... (2)

From (1) and (2),

$$\boxed{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$$

Remark : Similarly we can prove

$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C)$

$\Rightarrow x \in A$ or $(x \in B \cap C)$

$\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow x \in (A \cup B)$ and $x \in (A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$ ... (1)

Now let $y \in (A \cup B) \cap (A \cup C)$

$\Rightarrow y \in A \cup B$ and $y \in A \cup C$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \in A$ or $y \in C)$

$\Rightarrow y \in A$ or $(y \in B$ and $y \in C)$

$\Rightarrow y \in A$ or $(y \in B \cap C)$

$\Rightarrow y \in A \cup (B \cap C)$

$\Rightarrow (A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$ ... (2)

From (1) and (2),

$\boxed{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$

Remark : Similarly we can prove that

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

5) i) $(A \cup B)' = A' \cap B'$

$x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\Rightarrow (A \cup B)' \subset (A' \cap B')$ ... (1)

Now let, $y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$ ... (2)

$\Rightarrow (A' \cap B') \subset (A \cup B)'$

From (1) and (2),

$\boxed{(A \cup B)' = (A' \cap B')}$

ii) $(A \cap B)' = A' \cup B'$

Let $x \in (A \cap B)'$

$\Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$ ... (1)

$\Rightarrow (A \cap B)' \subset A' \cup B'$

Now let, $y \in A' \cup B'$

$\Rightarrow y \in A'$ or $y \in B'$

$\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\Rightarrow (A' \cup B') \subset (A \cap B)'$ ... (2)

From (1) and (2),

$\boxed{(A \cap B)' = A' \cup B'}$

6) i) $A \cup (A \cap B) = A$

Let $x \in A \cup (A \cap B)$

$\Rightarrow x \in A$ or $x \in (A \cap B)$

$\Rightarrow x \in A$ or ($x \in A$ and $x \in B$)

$\Rightarrow$ ($x \in A$ or $x \in A$) and ($x \in A$ or $x \in B$)

$\Rightarrow x \in A$ and $x \in (A \cup B)$

$\Rightarrow x \in A$ [as $A \subset A \cup B$]

$\Rightarrow A \cup (A \cap B) \subset A$ ... (1)

Now let, $y \in A$

$\Rightarrow y \in A \cup B$

$\Rightarrow y \in A$ and $y \in A \cup B$

$\Rightarrow y \in A$ and ($y \in A$ or $y \in B$)

$\Rightarrow$ ($y \in A$ or $y \in A$) and ($y \in A$ or $y \in B$)

$\Rightarrow y \in A$ or ($y \in A$ and $y \in B$)

$\Rightarrow y \in A$ or $y \in A \cap B$

$\Rightarrow y \in A \cup (A \cap B)$

$\Rightarrow A \subset A \cup (A \cap B)$ ... (2)

From (1) and (2),

$$\boxed{A \cup (A \cap B) = A}$$

Remark : As $A \cap B \subset A$ then $A \cup (A \cap B) = A$ as union of subset and super set = super set.

ii) $A \cap (A \cup B) = A$

Let $x \in A \cap (A \cup B)$

$\Rightarrow x \in A$ and $x \in (A \cup B)$

$\Rightarrow x \in A$ and ($x \in A$ or $x \in B$)

$\Rightarrow$ ($x \in A$ and $x \in A$) or ($x \in A$ and $x \in B$)

$\Rightarrow x \in A$ or $x \in A \cap B$

$\Rightarrow x \in A$ [as $A \cap B \subset A$]

$\Rightarrow A \cap (A \cup B) \subset A$ ... (1)

Now let, $y \in A$

$\Rightarrow y \in A$ or $y \in A \cap B$

$\Rightarrow$ ($y \in A$ and $y \in A$) or ($y \in A$ and $y \in B$)

$\Rightarrow y \in A$ and ($y \in A$ or $y \in B$)

$\Rightarrow y \in A$ and $y \in (A \cup B)$

$\Rightarrow y \in A \cap (A \cup B)$

$\Rightarrow A \subset A \cap (A \cup B)$ ... (2)

From (1) and (2),

$$\boxed{A \cap (A \cup B) = A}$$

Remark : As $A \subset A \cup B$ then $A \cap (A \cup B) = A$ as intersection of subset and super set is subset.

7) $(A')' = A$

Let $x \in (A')'$

$\Rightarrow x \notin A'$

$\Rightarrow x \in A$

$\Rightarrow (A')' \subset A$ ... (1)

Now let, $y \in A$

$\Rightarrow y \notin A'$

$\Rightarrow y \in (A')'$

$\Rightarrow A \subset (A')'$ ... (2)

From (1) and (2),

$$\boxed{(A')' = A}$$

1.13 Partitions of a Set

Let A be a set and $A_1, A_2, \ldots A_n$ are subsets of A then $A_1, A_2, \ldots A_n$ are called partitions of A if

i) $A_1 \cup A_2 \cup \ldots \cup A_n = A$

i.e. $\bigcup_{i=1}^{n} A_i = A$

and sets are pairwise disjoint.

i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$

Example

1. $A - B$, $B - A$ and $A \cap B$ are partitions of $A \cup B$. As $(A - B) \cup (B - A) \cup (A \cap B) =$ $A \cup B$ and $(A - B)$, $(B - A)$ and $A \cap B$ sets are pairwise disjoint.

1.14 Some Conclusions From Set Operations

Let X be the universal set, and A, B, C, D are subsets of X.

Then

1. $A \cup \emptyset = A$

   Hence $\emptyset$ i.e. empty set is identity set for 'union operation'.

2. $A \cap X = A$

   X i.e. universal set is identity set for 'intersection operation'.

3. $A \cup X = X$

   $A \subset X$ and $A \cup X = X$

   $\Rightarrow$ if $B \subset A$ then $A \cup B = A$ i.e. union of super set and subset is super set.

4. $A \cap \emptyset = \emptyset$

   $\emptyset \subset A$ and $A \cap \emptyset = \emptyset$

   $\Rightarrow$ if $B \subset A$, $A \cap B = B$ i.e. intersection of super set and subset is subset.

   Hence

5. If $B \subset A$, then $A \cap B = B$ and $A \cup B = A$

6. $A \subset A \cup B$ also $B \subset A \cup B$

7. $A \cap B \subset A$ also $A \cap B \subset B$

8. $A \cap B \subset A \subset A \cup B$ and $A \cap B \subset B \subset A \cup B$ i.e. in A and B sets. $A \cap B$ is the smallest set and $A \cup B$ is largest set.

   A contains $A \cap B$ and A is contained in $A \cup B$.

   Similarly B contains $A \cap B$ and B is contained in $A \cup B$.

9. As $A \cap B \subset A$

   $\Rightarrow$ $(A \cap B) \cup A = A$ [super set]

   and $(A \cap B) \cap A = (A \cap B)$ [subset]

10. As $A \subset A \cup B$

    $A \cap (A \cup B) = A$ [subset]

    $A \cup (A \cup B) = A \cup B$ [super set]

11. $(A - B) \cup (A \cap B) \cup (B - A) = A \cup B$

    and $(A - B)$, $(A \cap B)$, $(B - A)$ are pairwise disjoint.

i.e.

$$(A - B) \cap (A \cap B) = \emptyset$$

$$(A - B) \cap (B - A) = \emptyset$$

$$(A \cap B) \cap (B - A) = \emptyset$$

Hence $(A - B)$, $(A \cap B)$ and $(B - A)$ are partitions of $A \cup B$.

12. $A \cup A' = X$

    i.e. Union of A and its complement gives identity set for intersection operation.

13. $A \cap A' = \emptyset$

    i.e. Intersection of A and its complement gives identity set for union operation.

Remark :

Property 11 and 12 are very useful for Boolean algebra as in Boolean algebra these properties are formed in a way

$$a + a' = 1$$

i.e. sum of element and its complement gives multiplicative identity element and

$$a \cdot a' = 0$$

Multiplication of element and its complement gives additive identity element.

1.15 Cardinality of Finite Set

Cardinality of a set is number of distinct elements in the set.

Cardinality of a set A is denoted by n(A) or $|A|$.

Examples

1. $A = \{2, 3, 4, 5\}$

   n (A) or $|A| = 4$

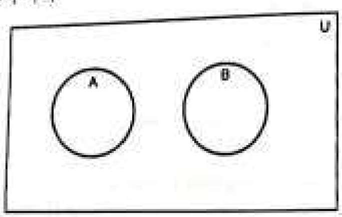
2. $A = \{x \,;\, 2 < x < 10\}$

   $A = \{3, 4, 5, 6, 7, 8, 9\}$

   $|A| = 7$

Theorem : If A and B are any two finite disjoint sets,

then $|A \cup B| = |A| + |B|$



Proof : If A and B are empty sets.

then $|A| = \varnothing, |B| = \varnothing$

$\Rightarrow \quad |A \cup B| = \varnothing$

$\Rightarrow \quad |A \cup B| = |A| + |B|$

Hence proved.

Now let A and B are non empty sets i.e. $|A| \neq \varnothing, |B| \neq \varnothing$.

This show A and B have some elements, say A has k_1 elements and B has k_2 elements then

$$A = \{a_1, a_2, a_3 \ldots a_{k_1}\}$$
$$B = \{b_1, b_2, b_3 \ldots b_{k_2}\}$$

As A and B are disjoint sets.

$$a_i \neq b_j \text{ for any } 1 \leq i \leq k_1, 1 \leq j \leq k_2$$

$\Rightarrow$ Number of elements in $A \cup B$ will be exactly $k_1 + k_2$ elements.

As $\quad A \cup B = \{a_1, a_2, \ldots a_{k_1}, b_1, b_2 \ldots b_{k_2}\}$

$\Rightarrow \quad |A \cup B| = |A| + |B|$

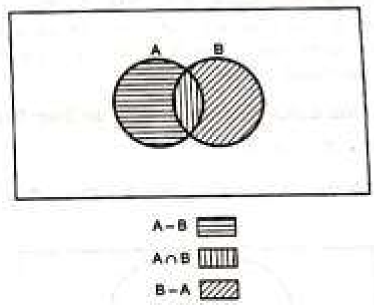
Hence proved.

Now let $A_1, A_2, A_3, \ldots A_k$ are finite disjoint sets then

$$|A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k| = |A_1| + |A_2| + \ldots |A_k|$$

$$\text{or } \left|\bigcup_{i=1}^{k} A_i\right| = \sum_{i=1}^{k} |A_i|$$

Now let A and B are two non empty sets. Common elements to A and B are $A \cap B$.



Number of elements which are present in A only (i.e. not in B) will be $|A - B|$. Number of elements in B only will be $|B - A|$ and number of elements which are common to both will be $|A \cap B|$.

$A \cup B$ is a union of $(A - B)$, $(B - A)$ and $(A \cap B)$ sets which are pairwise disjoint.

Hence $|A \cup B| = |A - B| + |A \cap B| + |B - A|$

From venn diagram it is clear that,

$$(A - B) \cup (A \cap B) = A$$

$$\Rightarrow \quad |A - B| + |A \cap B| = |A|$$

$$\text{or } |A - B| = |A| - |A \cap B|$$

$$\text{Similarly } (A \cap B) \cup (B - A) = B$$

$$|A \cap B| + |B - A| = |B|$$

$$\Rightarrow \quad |B - A| = B - |A \cap B|$$

$$\Rightarrow \quad |A \cup B| = |A - B| + |B - A| + |A \cap B|$$

$$= [|A| - |A \cap B|] + [|B| - |A \cap B|] + |A \cap B|$$

$$= |A| + |B| - |A \cap B|$$

Remark

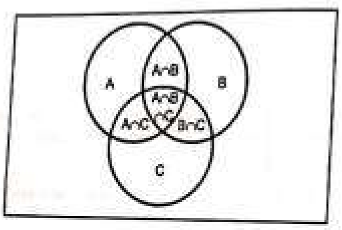
In the set $A \cup B$, to find the cardinality of $A \cup B$ i.e. $|A \cup B|$.

Number of elements of A and number of elements of B are included and number of elements common to both the sets A and B are excluded. Therefore this principle is known as Principle of Inclusion and Exclusion.

1.16 Principle of Inclusion and Exclusion for Two Sets

$$|A \cup B| = |A| + |B| - |A \cap B|$$

1.16.1 Principle of Inclusion and Exclusion for Three Sets A, B and C



For $|A \cup B \cup C|$

Let $B \cup C = D$

Then cardinality of $A \cup (B \cup C)$ will be cardinality of $A \cup D$.

i.e. $|A \cup D| = |A| + |D| - |A \cap D|$

Now $D = B \cup C$

Hence $|D| = |B \cup C|$

$= |B| + |C| - |B \cap C|$

So $|A \cup B \cup C| = |A| + [|B| + |C| - |B \cap C|] - |A \cap (B \cup C)|$

Now $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ using distributive property

$\Rightarrow$ $|A \cap (B \cup C)| = |(A \cap B) \cup (A \cap C)|$

$= |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$

$\Rightarrow$ $|A \cap (B \cup C)| = |A \cap B| + |A \cap C| - |(A \cap B) \cap (A \cap C)|$

$= |A \cap B| + |A \cap C| - |A \cap B \cap C|$

Hence $|A \cup B \cup C| = |A| + |B| + |C| - |B \cap C|$

$- [|A \cap B| + |A \cap C| - |A \cap B \cap C|]$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

1.16.2 Principle of Inclusion and Exclusion for Four Sets

Let A, B, C and D be four sets.

Then $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D|$

$- [|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|]$

$+ [|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|]$

$- |A \cap B \cap C \cap D|$

Proof : Let $B \cup C \cup D = G$

Then $|A \cup G| = |A| + |G| - |A \cap G|$

$|G| = |B \cup C \cup D|$

$= |B| + |C| + |D| - [|B \cap C| + |B \cap D| + |C \cap D|] + |B \cap C \cap D|$

and $|A \cap G| = |A \cap (B \cup C \cup D)|$

$= |(A \cap B) \cup (A \cap C) \cup (A \cap D)|$

$= |A \cap B| + |A \cap C| + |A \cap D|$

$- [|(A \cap B) \cap (A \cap C)| + |(A \cap B) \cap (A \cap D)| + |(A \cap C) \cap (A \cap D)|]$

$+ |(A \cap B) \cap (A \cap C) \cap (A \cap D)|$

$\Rightarrow$ $|A \cap G| = |A \cap B| + |A \cap C| + |A \cap D|$

$- [|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|] + |A \cap B \cap C \cap D|$

Substituting $|G| = |B \cup C \cup D|$ and $|A \cap G|$ in $|A \cup G|$.

Hence $|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - \begin{bmatrix} |A \cap B| + |A \cap C| + |A \cap D| \\ + |B \cap C| + |B \cap D| + |C \cap D| \end{bmatrix}$

$+ [|A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D|]$

$- |A \cap B \cap C \cap D|$

Remark

1. **For 3 sets :** a) Taking 1 set at a time, ${}^3C_1$ combinations can be formed, i.e. {A}, {B}, {C}.
   Taking 2 sets at a time, ${}^3C_2$ combinations can be formed, i.e. $\{A \cap B\}, \{B \cap C\}, \{A \cap C\}$.
   Taking 3 sets at a time, ${}^3C_3$ combinations can be formed, i.e. $\{A \cap B \cap C\}$.
   b) In inclusion, exclusion principle, one set at a time are included (i.e. added) two sets at a time are excluded (i.e. subtracted), three sets at a time are included.
2. **For 4 sets :** a) Number of 1 set at a time = ${}^4C_1$, i.e. {A}, {B}, {C}, {D}.
   Number of 2 sets at a time = ${}^4C_2$,
   i.e. $\{A \cap B\}, \{A \cap C\}, \{B \cap C\}, \{B \cap D\}, \{C \cap D\}, \{A \cap D\}$.
   Number of 3 sets at a time = ${}^4C_3$, i.e.
   $\{A \cap B \cap C\}, \{A \cap B \cap D\}, \{A \cap C \cap D\}, \{B \cap C \cap D\}$.
   And number of 4 sets at a time = ${}^4C_4$ i.e. $\{A \cap B \cap C \cap D\}$.
   b) One set at a time are included, two sets at a time are excluded, three sets at a time are included and four sets at a time are excluded.
3. Taking odd number of sets at a time are included and taking even number of sets at a time are excluded.

Theorem : Let $\{A_1, A_2, \ldots A_n\}$ be a finite collection of sets. Then

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i \le j \le n} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Proof : Proof is by induction on n.

The theorem is true for n = 2, 3 and 4. Let us assume the throrem for (n – 1) numbers of sets and prove it for n sets.

$$\begin{aligned} |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n| &= |(A_1 \cup A_2 \cup A_3 \ldots \cup A_{n-1}) \cup A_n| \\ &= |A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_{n-1}| + |A_n| \\ &\quad - |(A_1 \cup A_2 \cup A_3 \ldots \cup A_{n-1}) \cap A_n| \quad \ldots (1) \end{aligned}$$

By induction hypothesis

$$|A_1 \cup A_2 \cup \ldots \cup A_{n-1}| = \sum_{i=1}^{n-1} |A_i| - \sum_{1 \le i \le j \le n-1} |A_i \cap A_j| + \sum_{1 \le i \le j \le k \le n-1} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-2} |A_1 \cap A_2 \ldots \cap A_{n-1}| \quad \ldots (2)$$

Now $|(A_1 \cup A_2 \cup A_3 \ldots \cup A_{n-1}) \cap A_n| =$

$$|(A_1 \cap A_n) \cup (A_2 \cap A_n) \cup \ldots \cup (A_{n-1} \cap A_n)|$$

$$= \sum_{i=1}^{n-1} |A_i \cap A_n| - \sum_{1 \le i \le j \le n-1} |A_i \cap A_j \cap A_n| + \sum_{1 \le i \le j \le k \le n-1} |A_i \cap A_j \cap A_k \cap A_n| - \ldots + (-1)^{n-2} |A_1 \cap A_2 \cap \ldots \cap A_{n-1} \cap A_n| \quad \ldots (3)$$

Substituting equation (2) and (3) in equation (1), we obtain the equation

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j < n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \ldots + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots \cap A_n|$$

Example 1.11 *Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7 ? Also indicate how many are divisible by 3 or 5 but not by 7.*

Solution : Let A, B, C denote the set of integers from 1 to 250 divisible by 3 or 5 or 7.

$$|A| = \left[\frac{250}{3}\right] = 83$$

$$|B| = \left[\frac{250}{5}\right] = 50$$

$$|C| = \left[\frac{250}{7}\right] = 35$$

$$|A \cap B| = \left[\frac{250}{3 \times 5}\right] = 16$$

$$|A \cap C| = \left[\frac{250}{3 \times 7}\right] = 11$$

Remark

[a] denotes integer part of a

e.g. [83.33] = 83

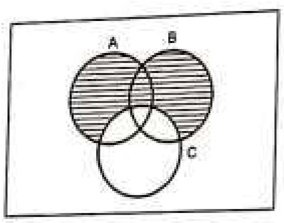
$$|B \cap C| = \left[\frac{250}{5 \times 7}\right] = 7$$

$$|A \cap B \cap C| = \left[\frac{250}{3 \times 5 \times 7}\right] = 1$$

$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 83 + 50 + 35 - [16 + 11 + 7] + 1$$

$$= 135$$



$|A \cup B \cup C| - |C|$

$|A|$ is number of integers divisible by 3.

$|B|$ number of integers divisible by 5.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 83 + 50 - 16$$

$$= 133 - 16$$

$$= 117$$

Number of elements divisible by 3 or 5 but not by 7 is

$$|A \cup B \cup C| - |C| = 135 - 35 = 100$$

Or

Number of elements divisible by 3 or 5 but not 7.

$|(A \cup B) - C|$ i.e.

$$= |A \cup B| - [|A \cap C| + |B \cap C| - |A \cap B \cap C|]$$

$$= 117 - [11 + 7 - 1]$$

$$= 117 - 17$$

$$= 100$$

Example 1.12 *How many integers between 1 to 2000 are divisible by 2, 3, 5 or 7.*

Solution : Suppose set A denotes the number of integers between 1 to 2000 divisible by 2.

Set B is the number of integers between 1 and 2000 divisible by 3.

Set C is the number of integers between 1 and 2000 divisible by 5.

Set D is the number of integers between 1 and 2000 divisible by 7.

$$|A| = \left[\frac{2000}{2}\right] = 1000$$

$$|B| = \left[\frac{2000}{3}\right] = 666$$

$$|C| = \left[\frac{2000}{5}\right] = 400$$

$$|D| = \left[\frac{2000}{7}\right] = 285$$

$$|A \cap B| = \left[\frac{2000}{2 \times 3}\right] = 333$$

$$|A \cap C| = \left[\frac{2000}{2 \times 5}\right] = 200$$

$$|A \cap D| = \left[\frac{2000}{2 \times 7}\right] = 142$$

$$|B \cap C| = \left[\frac{2000}{3 \times 5}\right] = 133$$

$$|B \cap D| = \left[\frac{2000}{3 \times 7}\right] = 95$$

$$|C \cap D| = \left[\frac{2000}{5 \times 7}\right] = 57$$

$$|A \cap B \cap C| = \left[\frac{2000}{2 \times 3 \times 5}\right] = 66$$

$$|A \cap B \cap D| = \left[\frac{2000}{2 \times 3 \times 7}\right] = 47$$

$$|A \cap C \cap D| = \left[\frac{2000}{2 \times 5 \times 7}\right] = 28$$

$$|B \cap C \cap D| = \left[\frac{2000}{3 \times 5 \times 7}\right] = 19$$

$$|A\cap B\cap C\cap D| = \left[\frac{2000}{2\times3\times5\times7}\right] = 9$$

Number of elements divisible by 2 or 3 or 5 or 7 are $|A\cup B\cup C\cup D|$.

From inclusion exclusion principle

$$|A\cup B\cup C\cup D| = |A|+|B|+|C|+|D|$$
$$-[|A\cap B|+|B\cap C|+|A\cap C|+|A\cap D|+|B\cap D|+|C\cap D|]$$
$$+[|A\cap B\cap C|+|A\cap B\cap D|+|A\cap C\cap D|+|B\cap C\cap D|]$$
$$-[|A\cap B\cap C\cap D|]$$

$$\therefore |A\cup B\cup C\cup D| = 1000+666+400+285-[333+200+142+133+95+57]$$
$$+[66+47+28+19]-9$$
$$= 2351-960+160-9$$
$$= 1542$$

Example 1.13 *Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology and 30 do not study any of the three subjects.*

a) Find the number of students studying all three subjects.

b) Find the number of students studying exactly one of the three subjects.

Solution : Let A, B, C denotes the set of students studying mathematics, physics and biology respectively.

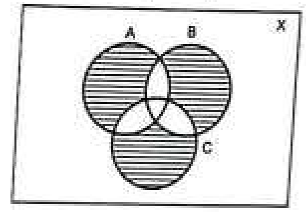
And
$$|X| = 100$$
$$|A| = 32$$
$$|B| = 20$$
$$|C| = 45$$
$$|A\cap C| = 15$$
$$|A\cap B| = 7$$
$$|B\cap C| = 10$$
$$|A'\cap B'\cap C'| = 30$$
$$|A'\cap B'\cap C'| = 100-|A\cup B\cup C|$$
Or
$$|A\cup B\cup C| = 100-30$$
$$= 70$$

a) $|A\cup B\cup C| = |A|+|B|+|C|-[|A\cap B|+|A\cap C|+|B\cap C|]+|A\cap B\cap C|$

$$70 = 32+20+45-[7+15+10]+|A\cap B\cap C|$$
$$70 = 97-[32]+|A\cap B\cap C|$$
$$70-65 = |A\cap B\cap C|$$
$$\Rightarrow |A\cap B\cap C| = 5$$

5 students study all 3 subjects.

b) Number of students studying exactly one subject.



Number of students studying only mathematics is

$$|A|-|A\cap B|-|A\cap C|+|A\cap B\cap C|$$
$$= 32-7-15+5$$
$$= 15$$

Number of students studying only physics is

$$|B|-|B\cap A|-|B\cap C|+|A\cap B\cap C|$$
$$= 20-7-10+5$$
$$= 8$$

Number of students studying only biology is

$$|C|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$
$$= 45-15-10+5$$
$$= 25$$

$\therefore$ Number of students studying exactly one subject

$$= 15+8+25 = 48$$

Example 1.16 *The 60,000 fans who attended the home coming football game bought up all the paraphernalia for their cars. Altogether 20,000 bumpers stickers, 36,000 window decals and 12,000 key rings were sold. We know that 52,000 fans bought at least one item and no one bought more than one of a given item. Also 6000 fans bought both decals and key rings, 9000 bought both decals and bumpers stickers and 5000 bought both key rings and bumper stickers.*

a) How many fans bought all three items ?

b) How many fans bought exactly one items ?

c) Someone questioned the accuracy of the total number of purchasers: 52000 (given that all the other numbers have been confirmed to be correct.) This person claimed the total number of purchasers to be either 60,000 or 44,000. How do you dispel the claim ?

Solution :

$|X| = 60{,}000$

Let $|A|$ be the number of fans who bought bumper sticker

$|A| = 20{,}000$

$|B|$ be the number of fans who bought window decals.

$|B| = 36{,}000$

$|C|$ be the number of fans bought key rings.

$|C| = 12{,}000$

$|A \cup B \cup C| = 52{,}000$

$|A \cap B| = 9000$

$|B \cap C| = 6000$

$|A \cap C| = 5000$

a) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$52{,}000 = 20{,}000 + 36{,}000 + 12{,}000 - [9000 + 6000 + 5000] + |A \cap B \cap C|$

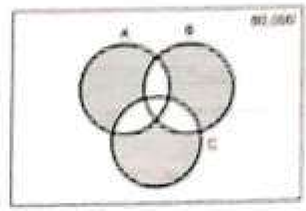
$52{,}000 = 68{,}000 - [20{,}000] + |A \cap B \cap C|$

$52{,}000 - 48{,}000 = |A \cap B \cap C|$

$\Rightarrow |A \cap B \cap C| = 4000$

Hence 4000 fans bought all the three items.

b)



Number of fans bought up only bumper stickers

$|A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$

$= 20{,}000 - 9000 - 5000 + 4000$

$= 10{,}000$

Number of fans bought up only window decals

$= |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$

$= 36{,}000 - 9000 - 6000 + 4000$

$= 25{,}000$

Number of fans bought up only key rings

$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$= 12{,}000 - 5000 - 6000 + 4000$

$= 5{,}000$

$\therefore$ Number of fans bought up exactly one item.

$= 10{,}000 + 25{,}000 + 5{,}000$

$= 40{,}000$

c) If $|A \cup B \cup C| = 60{,}000$

Then $60{,}000 = 20{,}000 + 36{,}000 + 12{,}000 - [9{,}000 + 6{,}000 + 5{,}000] + |A \cap B \cap C|$

$\Rightarrow |A \cap B \cap C| = 60{,}000 - 48{,}000$

$= 12{,}000$

which is not possible.

As $|A \cap B \cap C| \subset A \cap B$

Also $|A \cap B \cap C| \subset A \cap C$

And $|A \cap B \cap C| \subset B \cap C$

$\Rightarrow$ Number of elements in $|A \cap B \cap C|$

Cannot exceed number of elements in super sets.

If $|A \cup B \cup C| = 44{,}000$

Then $|A \cap B \cap C| = 44{,}000 - 48{,}000 = -4000$

which is again not possible.

Example 1.15 *A survey was conducted among 1000 people of these 595 are Democrats, 595 wear glasses, and 550 like icecream. 395 of them are Democrats who wear glasses, 350 of them are Democrats who like icecream, and 400 of them wear glasses and like icecream 250 of them are Democrats who wear glasses and like icecream. How many of them are not Democrats do not wear glasses, and do not like icecream ? How many of them are Democrats who do not wear glasses and do not like icecream ?*

Solution : $|X| = 1000$

Let $|A|$ be the number of Democrats.

$|A| = 595$

$|B|$ be the number of people who wear glasses.

$|B| = 595$

$|C|$ be the number of people who like icecream.

$|C| = 550$

$|A \cap B| = 395$

$|A \cap C| = 350$

$|B \cap C| = 400$

$|A \cap B \cap C| = 250$

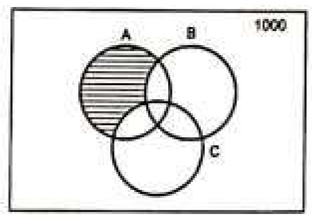
$$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|B \cap C|+|A \cap C|]+|A \cap B \cap C|$$

$$= 595 + 595 + 550 - [395 + 350 + 400] + 250$$

$|A \cup B \cup C| = 845$

845 people are either Democrats or wear glasses or icecream.

$\Rightarrow$ 1000 – 845 = 155 people are neither Democrats, nor wear glasses nor like icecream.



Number of people who are Democrats who do not like icecream and do not wear glasses.

$$= |A|-|A \cap B|-|A \cap C|+|A \cap B \cap C|$$

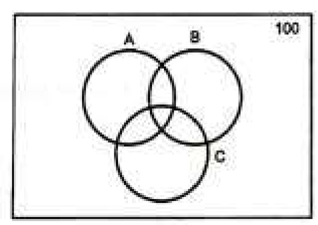
$$= 595 - [395 + 300] + 250$$

$$= 845 - 695$$

$$= 150$$

Example 1.16 *It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge. 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If some one claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ?*

Solution :



Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.

$$|A| = 60$$
$$|B| = 50$$
$$|C| = 70$$
$$|A \cap B| = 20$$
$$|A \cap C| = 30$$
$$|B \cap C| = 40$$
$$|A \cap B \cap C| = 20$$
$$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$$
$$= 60 + 50 + 70 - [20 + 30 + 40] + 20$$
$$= 110$$

which is not possible as $|A \cup B \cup C| \subset X$ and the number of elements in $|A \cup B \cup C|$ cannot exceed number of elements in the universal set X.

Example 1.17 *Among 200 students in a class, 104 students got an 'A', in first examination and 84 students got 'A' in second examination. If 68 students did not get an 'A' in either of the examination.*

i) How many students got 'A' in both the examination.

ii) If number of students who got an 'A' in the first examination is equal to that who got an 'A' in second examination. If the total number of students who got 'A' in exactly one examination is 160 and if 16 students did not get 'A' in either examination. Determine the number of students who got 'A' in first examination, those who got 'A' in second examination and number of students who got 'A' in both examinations.

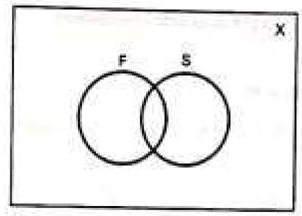
Solution : $X \rightarrow$ Universal set

$$|X| = 200$$

Let F denote the set of students who got an 'A' in first examination and S denote the set of students who got an 'A' in second examination.

68 students did not get A in either examination i.e.

$$|(F \cup S)'| = 68$$
$$\Rightarrow \quad |F \cup S| = |X| - |(F \cup S)'|$$
$$= 200 - 68$$
$$\Rightarrow \quad |F \cup S| = 132$$



i) Number of students who got A in both the examination will be $|F \cap S|$.

Now $\quad |F \cup S| = |F|+|S|-|F \cap S|$

$$132 = 104 + 84 - |F \cap S|$$
$$\therefore \quad |F \cap S| = 56$$

ii) $\quad |F| = |S|$

The total number of students who got A in exactly one examination is $|F-S| \cup |S-F|$ i.e. $F \oplus S$

and $\quad |F \oplus S| = 160$

16 students did not get A in either examination

$$\therefore \quad |(F \cup S)'| = 16$$
$$\Rightarrow \quad |F \cup S| = |X| - |(F \cup S)'|$$
$$\Rightarrow \quad |F \cup S| = 200 - 16 = 184$$

Now $\quad |F \cup S| = |F \oplus S| + |F \cap S|$

As $F \oplus S$ and $F \cap S$ are pairwise disjoint sets.

$$\Rightarrow \quad 184 = 160 + |F \cap S|$$
$$\Rightarrow \quad |F \cap S| = 184 - 160 = 24$$
$$\therefore \quad |F \cup S| = |F|+|S|-|F \cap S|$$
$$184 = 2|F| - 24$$
$$2|F| = 208$$
$$\Rightarrow \quad |F| = 104 \text{ and } |F| = |S|$$
$$\Rightarrow \quad |S| = 104$$

$\therefore$ Number of students who got A in first examination

$$= |F| - |F \cap S|$$

$$= 104 - 24 = 80$$

Similarly number of students who got A in second examination

$$= |S| - |F \cap S| = 104 - 24 = 80$$

Example 1.18 *Consider a set of integers 1 to 500. Find*

i) How many of these numbers are divisible by 3 or 5 or by 11 ?

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ?

Solution : Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

C denote numbers divisible by 11.

$|A|$ denotes cardinality of A similarly $|B|$ and $|C|$ denotes cardinality of B and C.

$$|A| = \left[\frac{500}{3}\right] = 166$$

$$|B| = \left[\frac{500}{5}\right] = 100$$

$$|C| = \left[\frac{500}{11}\right] = 45$$

$$|A \cap B| = \left[\frac{500}{3 \times 5}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{3 \times 11}\right] = 15$$

$$|B \cap C| = \left[\frac{500}{5 \times 11}\right] = 9$$

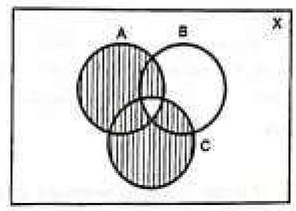
$$|A \cap B \cap C| = \left[\frac{500}{3 \times 5 \times 11}\right] = 3$$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$= 166 + 100 + 45 - [13 + 15 + 9] + 3$$

$$= 257$$

ii)



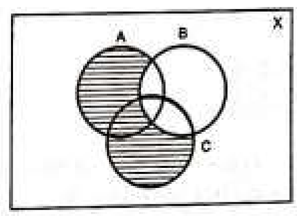
Number of integers divisible by 3 or by 11 but not by all 3, 5 and 11.

$$= |A \cup C| - |A \cap B \cap C|$$

$$= [|A| + |C| - |A \cap C|] - |A \cap B \cap C|$$

$$= 166 + 45 - 15 - 3 = 193$$

iii)



Number of integers divisible by 3 or 11 but not by 5.

$$= |A \cup B \cup C| - |B|$$

$$= 257 - 100 = 157$$

Example 1.19 *In the survey of 60 people, it was found that 25 read newsweek magazine, 26 read time, 26 read fortune. Also 9 read both newsweek and fortune, 11 read both newsweek and time, 8 read both time and fortune and 8 read no magazine at all.*

i) Find out the number of people who read all the three magazines.

ii) Fill in the correct numbers in all the regions of the Venn diagram.

iii) Determine number of people who reads exactly one magazine.

Solution : $X \rightarrow$ Universal set

$$|X| = 60$$

Let N denote the number of people who read newsweek magazine.

$\therefore \quad |N| = 25$

T denote the number of people who read time magazine.

$\therefore \quad |T| = 26$

F denote the number of people who read fortune magazine.

$\therefore \quad |F| = 26$

$|N \cap T| = 11$

$|N \cap F| = 9$

$|T \cap F| = 8$

$|N' \cap T' \cap F'| = 8$

Using DeMorgan's law

$|N' \cap T' \cap F'| = |X| - |N \cup T \cup F|$

$8 = 60 - |N \cup T \cup F|$

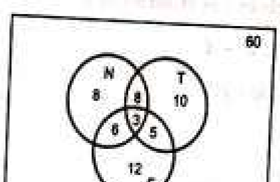
$\Rightarrow |N \cup T \cup F| = 52$

i) $|N \cup T \cup F| = |N| + |T| + |F| - [|N \cap T| + |N \cap F| + |T \cap F|] + |N \cap T \cap F|$

$\Rightarrow \quad 52 = 25 + 26 + 26 - [11 + 9 + 8] + |N \cap T \cap F|$

$\Rightarrow \quad |N \cap T \cap F| = 3$

ii)



Explanation :

As $|N \cap T \cap F| = 3$

and people who read N and T both = 11 which includes people of $|N \cap T \cap F|$ also.

Hence people who read only N and T but not F will be 11 – 3, hence 8 people read only N and T similarly $|T \cap F| = 8$.

Then people read only T and F = 8 – 3 i.e. 5.

and $\quad |N \cap F| = 9$

$\Rightarrow$ People read only N and F = 9 – 3 i.e. 6.

Also,

$|N| = 25$, which includes $|N \cap T|$, $|N \cap F|$ and $|N \cap T \cap F|$.

People who read only N $= |N| - [|N \cap T| + |N \cap F|] + |N \cap T \cap F|$

$= 25 - [11 + 9] + 3$

$\therefore$ People who read only N = 8 ...(1)

People who read only T $= |T| - [|N \cap T| + |T \cap F|] + |N \cap T \cap F|$

$= 26 - [11 + 8] + 3$

$\therefore$ People who read only T = 10 ... (2)

People who read only F $= |F| - [|N \cap F| + |T \cap F|] + |N \cap T \cap F|$

$= 26 - [9 + 8] + 3$

$\therefore$ People who read only F = 12 ... (3)

iii) Number of people who reads exactly one magazine

= Equation [(1) + (2) + (3)] = 8 + 10 + 12 = 30

Example 1.20 *Suppose that 100 out of 120 mathematics students at a college take at least one of the languages French, German and Russian. Also suppose*

65 study French, 20 study French and German

45 study German, 25 study French and Russian

42 study Russian, 15 study German and Russian

i) Find the number of students who study all the three languages.

ii) Fill in correct number of students in each region of Venn diagram.

iii) Determine the number K of students who study

a) Exactly one language.

b) Exactly two languages.

Solution : $X \rightarrow$ Universal set

$|X| = 120$

Let F be the number of students who study French. G be the number of students who study German and R be the number of students who study Russian.

$|F \cup G \cup R| = 100$

$|F| = 65, \quad |F \cap G| = 20$

$|G| = 45, \quad |F \cap R| = 25$

$|R| = 42, \quad |G \cap R| = 15$

i) $|F \cup G \cup R| = |F|+|G|+|R|-[|F \cap G|+|F \cap R|+|G \cap R|]+|F \cap G \cap R|$

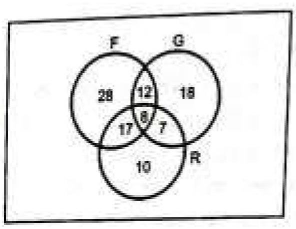
$100 = 65 + 45 + 42 - [20 + 25 + 15] + |F \cap G \cap R|$

$100 = 92 + |F \cap G \cap R|$

$\Rightarrow |F \cap G \cap R| = 8$

Hence 8 students study all three languages.

ii)



Explanation

As $|F \cap G \cap R| = 8$

Students who study F and G but not all 3 will be

$|F \cap G| - |F \cap G \cap R|$

$= 20 - 8 = 12$ ... (1)

Students who study F and R but not all 3 will be

$|F \cap R| - |F \cap G \cap R|$

$= 25 - 8 = 17$ ... (2)

Students who study G and R but not all 3 will be

$|G \cap R| - |F \cap G \cap R|$

$= 15 - 8 = 7$ ... (3)

Students who study only F but not G and not R.

$= |F|-[|F \cap G|+|F \cap R|]+|F \cap G \cap R|$

$= 65 - [20 + 25] + 8 = 28$ ... (4)

Students who study only G but not F and not R

$= |G|-[|F \cap G|+|G \cap R|]+|F \cap G \cap R|$

$= 45 - [20 + 15] + 8 = 18$ ... (5)

Students who study only R but not F and not G.

$= |R|-[|F \cap R|+|G \cap R|]+|F \cap G \cap R|$

$= 42 - [25 + 15] + 8 = 10$ ... (6)

iii) a) Number of students who study exactly one language

= Equation (4) + Equation (5) + Equation (6)

$= 28 + 18 + 10 = 56$

b) Number of students who study exactly two languages.

= Equation (1) + Equation (2) + Equation (3)

$= 12 + 17 + 7 = 36$

Example 1.21 *Out of the integers 1 to 1000.*

i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

ii) How many are not divisible by 5 and 7 but divisible by 3 ?

Solution : i) Let A denote numbers divisible by 3.

B denote numbers divisible by 5.

and C denote numbers divisible by 7.

$$|A| = \left[\frac{1000}{3}\right] = 333$$

$$|B| = \left[\frac{1000}{5}\right] = 200$$

$$|C| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{3 \times 5}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{3 \times 7}\right] = 47$$

$$|B \cap C| = \left[\frac{1000}{5 \times 7}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{3 \times 5 \times 7}\right] = 9$$

$$|A \cup B \cup C| = |A|+|B|+|C|-[|A \cap B|+|A \cap C|+|B \cap C|]+|A \cap B \cap C|$$

$$= 333 + 200 + 142 - [66 + 47 + 28] + 9$$

$$= 543$$

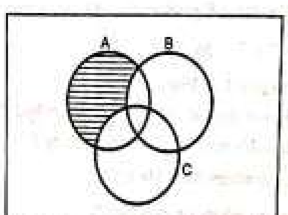
This show 543 numbers are divisible by 3 or 5 or 7. Hence numbers which are not divisible by 3, nor by 5, nor by 7.

$$= |A' \cap B' \cap C'|$$

$$= |X| - |A \cup B \cup C|$$

$$= 1000 - 543 = 457$$

ii)



Number of integers divisible by 3 but not by 5 and not by 7 is $|A \cap B' \cap C'|$.

$$|A \cap B' \cap C'| = |A \cap (B \cup C)'|$$

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 333 - [66 + 47] + 9 = 229$$

Example 1.22 *In the survey of 100 new cars, it is found that 60 had Air Conditioner (AC), 48 had power staring (PS), 44 had Power Windows (PW), 36 had AC + PW, 20 had PS + AC, 16 had PW + PS, 12 had all the three features. Find the number of cars that had : i) Only PW ii) PS and PW but not AC.*
iii) AC and PS but not PW.

Solution : $X \rightarrow$ Universal set

$$|X| = 100$$

Let A denote the number of AC in the cars,

$$|A| = 60$$

S denote the number of PS in the cars,

$$|S| = 48$$

and W denote the number of PW in the cars,

$$|W| = 44$$

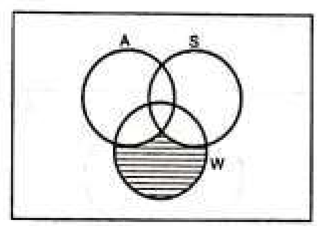
$$|A \cap W| = 36$$

$$|A \cap S| = 20$$

$$|S \cap W| = 16$$

$$|A \cap S \cap W| = 12$$

i)



Number of cars that had only power windows = $|W \cap A' \cap S'|$

$$|W \cap A' \cap S'| = W \cap (A \cup S)'$$

$$= W - (A \cup S)$$

$$|W - (A \cup S)| = |W| - [(W \cap A) \cup (W \cap S)]$$

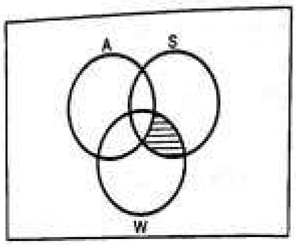
Now

$$|(W \cap A) \cup (W \cap S)| = |W \cap A| + |W \cap S| - |W \cap A \cap S|$$

$$\therefore \quad |W \cap A' \cap S'| = |W| - [|W \cap A| + |W \cap S|] + |W \cap S \cap A|$$

$$= 44 - [36 + 16] + 12 = 4$$

ii)



Number of cars that had PS and PW but not AC.

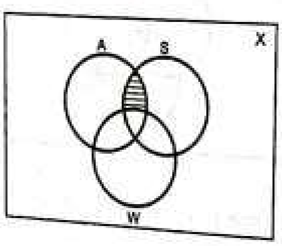
$$|S \cap W \cap A'| = |S \cap W| - |A \cap S \cap W|$$

$$= 16 - 12 = 4$$

iii) Number of cars that had AC and PS but not PW.

$$|A \cap S \cap W'| = |A \cap S| - |A \cap S \cap W|$$

$$= 20 - 12 = 8$$



Example 1.23 *Among 100 students, 32 study mathematics, 20 study physics, 45 study biology, 15 study mathematics and biology, 7 study mathematics and physics, 10 study physics and biology, 30 do not study any of the three subjects :*

i) Find the number of students studying all the three subjects.

ii) Find the number of students studying exactly one of the three subjects.

Solution : Let X denote universal set. M denote set of students studying mathematics.

P denote set of students studying physics and B denote set of students studying biology.

Then $|X| = 100, \; |M| = 32, \; |P| = 20, \; |B| = 45,$

$$|M \cap B| = 15, \; |M \cap P| = 7, \; |P \cap B| = 10$$

$$|M' \cap B' \cap P'| = 30 \Rightarrow |(M \cup B \cup P)'| = 30$$

or $|M \cup B \cup P| = |X| - |M \cup B \cup P|$

$$= 100 - 30 = 70$$

i) $|M \cup B \cup P| = |M| + |B| + |P| - [|M \cap B| + |M \cap P| + |B \cap P|] + |M \cap B \cap P|$

$$70 = 32 + 45 + 20 - [15 + 7 + 10] + |M \cap B \cap P|$$

$\Rightarrow |M \cap B \cap P| = 5$

ii) Number of students studying mathematics but not biology and not physics.

$$= |M| - [|M \cap B| + |M \cap P|] + |M \cap B \cap P|$$

$$= 32 - [15 + 7] + 5 = 15 \quad \ldots (1)$$

Number of students studying biology but not mathematics and not physics.

$$= |B| - [|M \cap B| + |P \cap B|] + |M \cap B \cap P|$$

$$= 45 - [15 + 10] + 5 = 25 \quad \ldots (2)$$

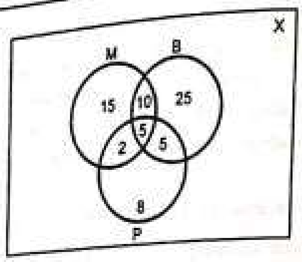
Number of students studying physics but not mathematics and not biology.

$$= |P| - [|M \cap P| + |B \cap P|] + |M \cap B \cap P|$$

$$= 20 - [7 + 10] + 5 = 8 \quad \ldots (3)$$

Hence number of students studying exactly one subject.

$$= 15 + 25 + 8 = 48$$



Example 1.24 *In the class of 55 students the number of studying different subjects are as given below : Maths 23, Physics 24, Chemistry 19, Maths + Physics 12, Maths + Chemistry 9, Physics + Chemistry 7, all three subjects 4.*

Find the number of students who have taken :

i) At least one subject ii) Exactly one subject iii) Exactly two subjects.

Solution : Let X denote universal set.

Let M denote set of students studying mathematics.

P denote set of students studying physics

and C denote set of students studying chemistry.

Then $|X| = 55, |M| = 23, |P| = 24, |C| = 19, |M \cap P| = 12,$

$|M \cap C| = 9, |P \cap C| = 7$

$|M \cap P \cap C| = 4$

i) $|M \cup P \cup C| = |M| + |P| + |C| - [|M \cap P| + |M \cap C| + |P \cap C|] + |M \cap P \cap C|$

$= 23 + 24 + 19 - [12 + 9 + 7] + 4 = 42$

ii) Number of students studying mathematics but not physics and not chemistry.

$= |M| - [|M \cap P| + |M \cap C|] + |M \cap P \cap C|$

$= 23 - [12 + 9] + 4 = 6$ ... (1)

Number of students studying physics but not mathematics and not chemistry.

$= |P| - [|M \cap P| + |P \cap C|] + |M \cap P \cap C|$

$= 24 - [12 + 7] + 4 = 9$ ... (2)

Number of students studying chemistry but not mathematics and not physics.

$= |C| - [|M \cap C| + |P \cap C|] + |M \cap P \cap C|$

$= 19 - [9 + 7] + 4 = 7$ ... (3)

Hence number of students studying exactly one subject = 6 + 9 + 7 = 22

iii) Number of students studying mathematics and physics but not chemistry

$= |M \cap P| - |M \cap P \cap C|$

$= 12 - 4 = 8$ ... (4)

Number of students studying physics and chemistry but not mathematics

$= |P \cap C| - |M \cap P \cap C|$

$= 7 - 4 = 3$ ... (5)

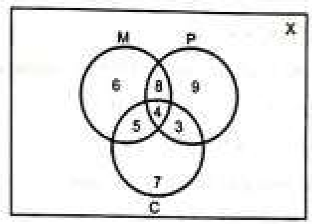
Number of students studying mathematics and chemistry but not physics

$= |M \cap C| - |M \cap P \cap C|$

$= 9 - 4 = 5$ ... (6)

Hence number of students studying exactly two subjects

$= 8 + 3 + 5 = 16$



Example 1.25 *One hundred sportsmen were asked whether they play which game : cricket, hockey, football. The results were :*

45 play cricket, 38 play hockey, 21 play football, 18 play cricket and hockey, 9 play cricket and football, 4 play football and hockey and 23 play none of these.

Draw a Venn diagram that will show the result of the survey and determine the number of sportsmen who play :

1) Exactly one of the games

2) Exactly two of the games.

Solution : Let X denote universal set.

Let 'C' denote number of sportsmen who play cricket.

'H' denote number of sportsmen who play hockey.

and 'F' denote number of sportsmen who play football.

$$|X| = 100,\ |C| = 45,\ |H| = 38,\ |F| = 21,\ |C \cap H| = 18,$$

$$|C \cap F| = 9,\ |F \cap H| = 4,\ |C' \cap H' \cap F'| = 23$$

$$|C' \cap H' \cap F'| = |(C \cup H \cup F)'|$$

$$|C \cup H \cup F| = |X| - |(C \cup H \cup F)'| = 100 - 23 = 77$$

$$|C \cup H \cup F| = |C| + |H| + |F| - [|C \cap H| + |C \cap F| + |H \cap F|] + |C \cap H \cap F|$$

$$|C \cup H \cup F| = |C| + |H| + |F| - [|C \cap H| + |C \cap F| + |H \cap F|] + |C \cap H \cap F|$$

$$77 = 45 + 38 + 21 - [18 + 9 + 4] + |C \cap H \cap F|$$

$$\Rightarrow \quad |C \cap H \cap F| = 4$$

i) Number of sportsmen who play only cricket but not hockey and not football.

$$= |C| - [|C \cap H| + |C \cap F|] + |C \cap H \cap F|$$

$$= 45 - [18 + 9] + 4 = 22 \qquad \text{... (1)}$$

Number of sportsmen who play only hockey but not cricket and not football.

$$= |H| - [|C \cap H| + |H \cap F|] + |C \cap H \cap F|$$

$$= 38 - [18 + 4] + 4 = 20 \qquad \text{... (2)}$$

Number of sportsmen who play only football but not cricket and not hockey.

$$= |F| - [|H \cap F| + |C \cap F|] + |C \cap H \cap F|$$

$$= 21 - [4 + 9] + 4 = 12$$

$\therefore$ Number of sportsmen who play exactly one game.

$$= 22 + 20 + 12 = 54$$

ii) Number of sportsmen who play cricket and hockey but not football.

$$= |C \cap H| - |C \cap H \cap F|$$

$$= 18 - 4 = 14 \qquad \text{... (4)}$$

Number of sportsmen who play cricket and football but not hockey.

$$= |C \cap F| - |C \cap H \cap F|$$

$$= 9 - 4 = 5 \qquad \text{... (5)}$$

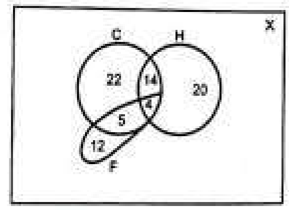
Number of sportmen who play hockey and football but not cricket.

$$= |H \cap F| - |C \cap H \cap F|$$

$$= 4 - 4 = 0 \qquad \text{... (6)}$$

Hence number of sportsmen who play exactly two games.

$$= 14 + 5 + 0 = 19$$



Example 1.26 *100 of the 120 engineering students in a college take part in atleast one of the activity group discussion, debate and quiz.*

Also : 65 participate in group discussion, 45 participate in debate, 42 participate in quiz, 20 participate in group discussion and debate 25 participate in group discussion and quiz 15 participate in debate and quiz. Find the number of students who participate in : i) All the three activities ii) Exactly one of the activities.

Solution : Let X denote universal set

$$|X| = 120$$

Let A denote number of students taking part in group discussion.

B denote number of students taking part in debate

and C denote number of students taking part in quiz.

$$|A \cup B \cup C| = 100,\ |A| = 65,\ |B| = 45,\ |C| = 42,$$

$$|A \cap B| = 20,\ |A \cap C| = 25,\ |B \cap C| = 15.$$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$100 = 65 + 45 + 42 - [20 + 25 + 15] + |A \cap B \cap C|$$

$$\Rightarrow |A \cap B \cap C| = 8$$

ii) Number of students taking part in group discussion but not in debate and not in quiz.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C| \qquad \text{... (1)}$$

$$= 65 - [20 + 25] + 8 = 28$$

Number of students taking part in debate but not in group discussion and not in quiz.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

$$= 45 - [20 + 15] + 8 = \mathbf{18}$$

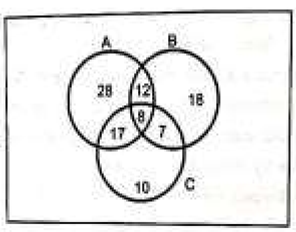
Number of students taking part in quiz but not in group discussion and not in debate.

$$= |C| - [|A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 42 - [25 + 15] + 8 = \mathbf{10}$$

Hence number of students doing exactly one activity

$$= 28 + 18 + 10 = \mathbf{56}$$



Example 1.27 *It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also know that 37 knew 'C' and COBOL, 28 'C' and PASCAL and 25 PASCAL and COBOL. 7 students however knew none of the languages. Find*

i) How many knew all the three languages ?

ii) How many knew exactly two languages ?

iii) How many knew exactly one language ?

Solution : Let A denote the set of students who know COBOL.

B denote the set of students who know 'C'.

and C denote the set of students who know PASCAL.

and X denote universal set.

Then $|X| = 80$, $|A| = 50$, $|B| = 55$, $|C| = 46$

$|A \cap B| = 37$, $|B \cap C| = 28$, $|A \cap C| = 25$

$|A' \cap B' \cap C'| = 7$

Hence $|(A \cup B \cup C)'| = 7$

Also $\quad |A \cup B \cup C| = |X| - |(A \cup B \cup C)'|$

Hence $\quad |A \cup B \cup C| = 80 - 7 = 73$

i) $\quad |A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$73 = 50 + 55 + 46 - [37 + 28 + 25] + |A \cap B \cap C|$$

$\Rightarrow \quad |A \cap B \cap C| = \mathbf{12}$

ii) Number of students who know only COBOL and 'C' but not PASCAL.

$$= |A \cap B| - |A \cap B \cap C|$$

$$= 37 - 12 = \mathbf{25} \qquad \text{... (1)}$$

Number of students who know only COBOL and PASCAL but not 'C'.

$$= |A \cap C| - |A \cap B \cap C|$$

$$= 25 - 12 = \mathbf{13} \qquad \text{... (2)}$$

Number of students who know only 'C' and PASCAL but not COBOL.

$$= |B \cap C| - |A \cap B \cap C|$$

$$= 28 - 12 = \mathbf{16} \qquad \text{... (3)}$$

Hence number of students who know exactly two languages.

$$= 25 + 13 + 16 = \mathbf{54}$$

iii) Number of students who know only COBOL but not 'C' and not PASCAL.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 50 - [37 + 25] + 12 = \mathbf{0} \qquad \text{... (4)}$$

Number of students who know only 'C' but not COBOL and not PASCAL.

$$= |B| - [|A \cap B| + |B \cap C|] + |A \cap B \cap C|$$

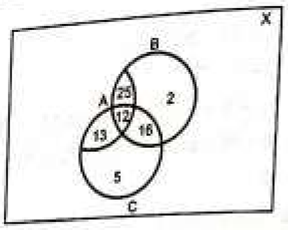
$$= 55 - [37 + 28] + 12 = \mathbf{2} \qquad \text{... (5)}$$

Number of students who know only PASCAL but not COBOL and not 'C'.

$$= |C| - [|A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

$$= 46 - [25 + 28] + 12 = \mathbf{5} \qquad \text{... (6)}$$

Hence number of students who know only one language = 0 + 2 + 5 = **7**



Example 1.28 *In a survey of 500 television watchers produced the following information 285 watch football, 195 watch hockey, 115 watch basketball, 45 watch football and basketball, 70 watch football and hockey, 50 watch hockey and basketball and 50 do not watch any of the three games.*

i) How many people in the survey watch all three games ?

ii) How many people watch exactly one game ?

Solution : Let F denote the set of people who watch football.

H denote the set of people who watch hockey and B denote the set of people who watch basketball.

$X \rightarrow$ Denote universal set

Then $|X| = 500, |F| = 285, |H| = 195, |B| = 115.$

$|F \cap B| = 45, |F \cap H| = 70, |H \cap B| = 50$

$|F' \cap B' \cap H'| = 50$

i.e. $|(F \cup B \cup H)'| = 50$

$\Rightarrow$ $|F \cup B \cup H| = |X| - |(F \cup B \cup H)'|$

$= 500 - 50 = 450$

i) Number of people who watch all the three games $= |F \cap H \cap B|$.

$|F \cup H \cup B| = |F| + |H| + |B| - [|F \cap H| + |F \cap B| + |H \cap B|] + |F \cap H \cap B|$

$450 = 285 + 195 + 115 - [70 + 45 + 50] + |F \cap H \cap B|$

$|F \cap H \cap B| = 20$

ii) Number of people who watch only football.

$= |F| - [|F \cap B| + |F \cap H|] + |F \cap B \cap H|$

$= 285 - [45 + 70] + 20 = 190$ ... (1)

Number of people who watch only hockey.

$= |H| - [|H \cap B| + |H \cap F|] + |F \cap B \cap H|$

$= 195 - [50 + 70] + 20 = 95$ ... (2)

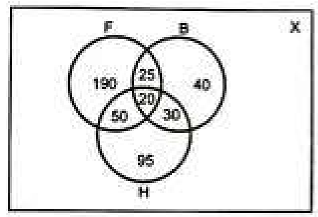
Number of people who watch only basketball.

$= |B| - [|B \cap H| + |B \cap F|] + |F \cap B \cap H|$

$= 115 - [50 + 45] + 20 = 40$ ... (3)

$\therefore$ Number of people who watch exactly one game.

$= 190 + 95 + 45 = 325$



Example 1.29 *A survey has been taken on methods of computer travel. Each respondent was asked to check bus, train or automobile as a major method of travelling to work. More than one answer was permitted. The result reported were as follows :*

Bus-30 people, train-35 people, automobile-100 people, bus and train-15 people, bus and automobile-15 people, train and automobile-20 people and all three methods-5 people. How many people completed a survey form ?

Solution : Let B denote set of people travel by bus. Let T denote set of people travel by train and let A denote set of people travel by automobile.

Then $|B| = 30, |T| = 35, |A| = 100, |B \cap T| = 15,$

$|B \cap A| = 15, |T \cap A| = 20, |B \cap T \cap A| = 5$

Number of people completed a survey form,

$= |B \cup T \cup A|$

$= |B| + |T| + |A| - [|B \cap T| + |T \cap A| + |B \cap A|] + |B \cap T \cap A|$

$= 30 + 35 + 100 - [15 + 15 + 20] + 5 = 120$

□□□

2 Functions

Syllabus

Introduction & definition, Co-domain, range, image, value of a function; Examples, surjective, injective, bijective; examples; Composition of functions, examples; Inverse function, Identity map, condition of a function to be invertible, examples; Inverse of composite functions, Properties of Composition of functions.

Contents

2.1 Functions or Mappings

Introduction

Let A be the set of 30 students and B be the set of 40 chair in a class room. The correspondence between the set A and set B is "student sitting on a chair."

Then the correnpondence f from A to B is a function or a mapping if and only if

1. Every student is sitting on a chair.
2. No student is sitting on two different chairs.

If these conditions are satisfied, f is called a function or mapping and it is denoted by $f : A \to B$.

Remark

1. If one student is standing, then f cannot be a function.
2. If one student is occupying two different chairs still f can not be a function i.e. f cannot be of the type one element is corresponding to many element.

2.1.1 Definition of Functions

Let A and B be two non-empty sets. Then a function or mapping f from the set A to the set B is a rule which assigns to each element $a \in A$ to unique element $b \in B$.

We say that f maps element a of set A to element b of set B and that f maps set A to set B.

The notation denote that f maps a to b is

$f(a) = b$ or $(a, b) \in f$.

[Remark : f is well defined if $f(a_1) = b$ and $f(a_1) = c$]

$\Rightarrow b = c$

2.1.2 Representation by Diagram

Let the interiors of the two closed areas represent the sets A and B.

Let $a_1, a_2, a_3 ...$ be the elements of A and $b_1, b_2, b_3, ...$ be the elements of B.

The mapping or function $f : A \to B$ is represented by means of arc of lines joining the points representing the elements of A to the elements of B.

1. Every $a \in A$ is joined to some $b \in B$.
2. Two or more points in A may be joined to the same point in B (as, a_1, a_2 are joined to b_2 in the figure 2.1.)
3. For mapping, two or more points of B cannot be joined to the same point in A.

Examples :

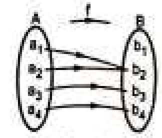


Fig. 2.1

1. Let $S_1 = \{1, 2, 3\}$

$S_2 = \{p, q, r, S\}$

For figure 2.2,

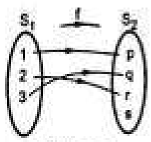


Fig. 2.2

$f : S_1 \to S_2$ is a function it can also be denoted by $S_1 \xrightarrow{f} S_2$

2. $A = \{x, y, z\}$, $B = \{1, 2, 3, 4\}$

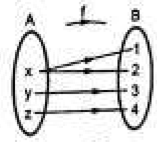


Fig. 2.3

For the figure 2.3,

f is not a function from A to B as x in A is associated with two different elements of B.

3. $A = \{x, y, z\}$, $B = \{1, 2, 3, 4\}$

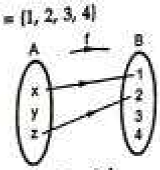


Fig. 2.4

For the figure 2.4,

f is not a function as y from set A is not in correspondence with any element of set B.

4. $A = \{x, y, z, t\}$, $B = \{1, 2, 3, 4, 5, 6\}$

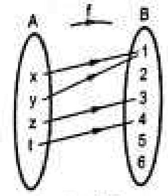


Fig. 2.5

For the figure 2.5,

f is function from A to B as every element of set A is in correspondence with unique element of set B.

[**Remark** : $f : A \to B$, f is a function from A to B, then two elements of A may correspond with one element of set B. Also B may have elements with which no element of A is corresponding].

2.1.3 Image of Function f

If f is a function from A to B, i.e. $f : A \to B$

then for $f(a) = b$, element b of B is called f image of element a of A and element a is called preimage of b.

2.1.4 Domain, Co-domain

If f is a function from A to B.

i.e. $f : A \to B$. Then set A is known as domain set, and set B is known as co-domain set.

2.1.5 Range of a Function

Range of (f) $= \{b : b \in B \text{ and } f(a) = b \text{ for some } a \in A\}$.

In other words, range of f is the set of all images of the elements of A under f.

Example 1 : $A = \{x, y, z, t\}$, $B = \{1, 2, 3, 4, 5\}$

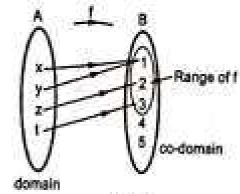


Fig. 2.6

$$f(x) = 1$$
$$f(y) = 1$$
$$f(z) = 2$$
$$f(t) = 3$$

A is known as domain set B is known as co-domain set.

Range of f is a subset of co-domain set.

Range of $f = \{1, 2, 3\}$ i.e. range of f is set of those elements of B which are associated with some elements of domain set A.

Elements of domain set are known as pre-image of f and elements of range are known as f-image of set A.

2.2 Function as Set of Ordered Pairs

If A and B are any two non-empty sets, then a function f from A to B is a subset of $A \times B$.

Satisfying the following conditions.

i) $\forall a \in A, (a, b) \in f$ for some $b \in B$

ii) $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$.

The first condition ensures that every element $a \in A$ is associated with some element $b \in B$. Thus each element in A will have image. Second condition ensures that elements are not having two different images i.e. each element $a \in A$ have unique image $b \in B$.

Example 1

Let

$$A = \{1, 2, 3, 4\}, \quad B = \{b_1, b_2, b_3\}$$
$$f = \{(1, b_1), (1, b_2), (2, b_3), (3, b_3), (4, b_2)\}$$

is not a function as 1 is associated with two different elements b_1 and b_2 of B.

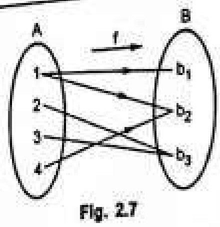


Fig. 2.7

Example 2

$$A = \{1, 2, 3, 4\} \quad B = \{b_1, b_2, b_3\},$$

$$f = \{(1, b_1), (2, b_1), (3, b_3)\}$$

is not a function as 4 from A is not associated with any element of B.

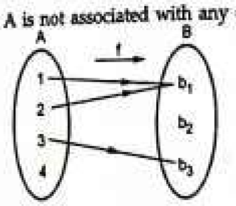


Fig. 2.8

Example 2.1 *If $A = \{x, y, z\}$, $B = \{a, b, c, d\}$, decide whether or not the following are functions from A to B. If they are functions, give the range of each; if not, tell why ?*

a) $f = \{(x, a), (y, b), (z, c)\}$ b) $f = \{(x, a), (y, c), (z, b), (x, c)\}$

c) $f = \{(x, d), (y, b)\}$ d) $f = (x, a), (y, b), (z, d)\}$

e) $f = \{(y, a), (y, b), (y, c), (y, d)\}$ f) $f = \{(x, b), (y, c), (z, d)\}$

Solution :

a) f is a function from A to B and range of f = {a, b, c}.

b) f is not a function as $x \in A$ have two different images a and c, i.e. image of x is not unique.

c) f in not a function as $z \in A$ does not correspond to any element of B, i.e. z has no image.

d) f is a function and range of f = {a, b, d}.

e) f is not a function as image of y is not unique also x and z have no images.

f) f is a function and range of f = {b, c ,d}

2.3 Equality of Two Functions

Two functions f and g from set A to set B are said to be equal functions iff $f(x) = g(x)$, $\forall x \in A$. If $\exists$ at least one element $y \in A$ s.t. $f(y) \neq g(y)$ then mappings f and g are not equal mappings.

e.g. $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

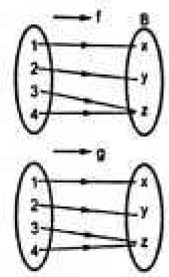


Fig. 2.9

then $\forall a \in A$

$$f(a) = g(a)$$

Hence f and g mappings are same.

2.4 Identity Mapping

Let A be any set and f be any function defined on set A.

i.e. $f : A \to A$

and $f(x) = x, \forall x \in A$

then f mapping is known as identity mapping or identity function and generally denoted by I_A i.e. identity mapping on set A.

e.g.

Let $A = \{x, y, z\}$

then

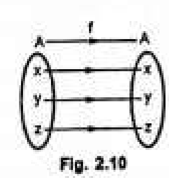


Fig. 2.10

f is identity mapping I_A

2.5 Constant Function

The function defined from set A to set B s.t. $f(a) = b$, $\forall a \in A$ then f is called constant function.

In order to prove that f is a constant function, it is enough to show that range of f consists of only one element.

Example 1 : $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

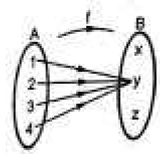


Fig. 2.11

$\forall\, a \in A$

$f(a) = y$

i.e. $f(1) = y$, $f(2) = y$, $f(3) = y$ and $f(4) = y$

Hence f is a constant function and range of $f = \{y\}$.

2.6 Types of Function or Mappings

2.6.1 One to One Mapping or Injective Mapping

A function $f : A \to B$ is said to be one to one mapping or one to one correspondence or injective function or univalent function.

if $\forall\, a_1, a_2 \in A$

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

or if $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$

Thus under an injective function an element of B can have at the most one element of A as preimage.

Example 1 $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, t, q\}$

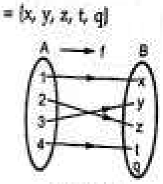


Fig. 2.12

$f : A \to B$ is one to one mapping.

Remark : In one to one mapping (abbreviated as 1 - 1) different elements of A have different f - images in B if $f : A \to B$.

or same elements of B have same preimages in A.

If f is an one to one function from A to B, then we write

$f : A \xrightarrow[1-1]{} B$.

2.6.2 Many to One Mapping

Let $f : A \to B$

then f is many to one function if $a_1 \neq a_2$ but $f(a_1) = f(a_2)$ i.e. f is many to one function if two (or more than two) distinct elements of A have the same f-images in B.

Example 1 : $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, t\}$

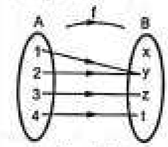


Fig. 2.13

$1 \neq 2$ but $f(1) = f(2)$

so f is many to one function.

2.6.3 Into Mapping

If $f : A \to B$

then f is into mapping from $A \to B$ if $\exists$ at least one element in B which is not f-image of any element of A. In this case range of f is a proper subset of co-domain set.

i.e. $f(A) \subset B$.

Thus in 'into' mapping at least one element of the co-domain B is left uncovered by f-images of the domain A.

Example 1 $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, t\}$

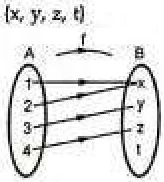


Fig. 2.14

$t \in B$ is an element which is left uncovered by the f-images of the domain A.

Hence f is an 'into' mapping.

2.6.4 Onto Mapping or Surjective Mapping

A function $f : A \rightarrow B$ is said to be onto mapping if each element of B is the f image of at least one element of set A.

In this case range of f is co-domain set B i.e. $f(A) = B$

Example 1 $A = \{1, 2, 3, 4\}$, $B = \{x, y, z\}$

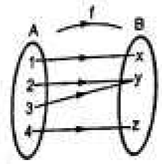


Fig. 2.15

Every element of B is f image of at least one element of A hence f is 'onto' function.

Hence range of f = co-domain set B.

Onto mapping is also known as **surjective** function (or a surjection) if f is an 'onto' function from A to B then, we write $f : A \xrightarrow{\text{onto}} B$.

2.6.5 Many One Into Mapping

If $f : A \rightarrow B$ is many one mapping and also f is into mapping then f is said to be many one into mapping.

Hence f is many one into mapping.

if 1) $f(a_1) = f(a_2)$ but $a_1 \neq a_2$.

2) $\exists$ at least one element in set B which is not the f-image of any element of set A (or range of f is proper subset of co-domain set B).

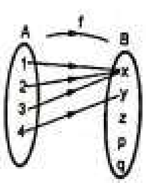


Fig. 2.16

Example 1 $A = \{1, 2, 3, 4\}$, $B = \{x, y, z, p, q\}$

$f(1) = f(2) = f(3)$

but $1 \neq 2 \neq 3$

So f is many one function. Also $\exists$ z, p, q in set B which are not f-image of any element of set A. Hence f is into mapping therefore f is many one into mapping.

2.6.6 Many One Onto Mapping

If $f : A \rightarrow B$ is many one mapping and also f is onto mapping then f is said to be many one onto mapping.

Hence f is many one onto mapping if

1) For some $a_1 \neq a_2 \Rightarrow f(a_1) = f(a_2)$

or

$f(a_1) = f(a_2)$ but $a_1 \neq a_2$

2) Every element of set B is f-image of at least one element of set A

i.e. range of f = co-domain set B

or $f(A) = B$

Example 1 $A = \{2, 4, 6, 8\}$, $B = \{x, y, z\}$

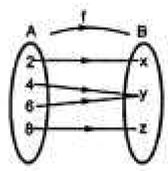


Fig. 2.17

f is a many one mapping, f is also onto mapping so f is many one onto mapping.

2.6.7 One One Into Mapping

A mapping $f : A \rightarrow B$ is one one into mapping if f is both one to one and also into mapping.

Hence f is one one into mapping,

if i) $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$, $\forall a_1, a_2 \in A$.

ii) $\exists$ at least one element in set B which is not the f-image of any element of set A (or range of f is a proper subset of co-domain set B).

Example 1 $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d, e, g\}$

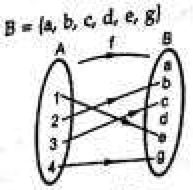


Fig. 2.18

f is one one as $1 \neq 3 \Rightarrow f(1) \neq f(3)$

$f(1) = e,\ f(3) = b$

$e \neq b$

Also range of $f = \{b, c, e, g\}$

i.e. range of f is proper subset of B. $\exists$ a, d $\in$ B which are not f-images of any element of A.

2.6.8 One One Onto Mapping or Bijective Mapping

A mapping $f : A \rightarrow B$ which is **one one** as well as **onto** is called one one onto mapping or bijection.

Hence f is one one onto if

i) $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

(or $a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2)$)

ii) Every element of set B is f-image of at least one element of set A.

i.e. range of f = co-domain set B.

or $f(A) = B$

or $\forall\, b \in B\ \exists\, a \in A$

s.t. $f(a) = b$

Example 1 $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

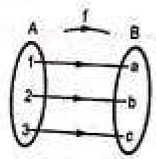


Fig. 2.19

f is one one and onto function.

2.7 Inverse Function (Mapping)

If $f : A \rightarrow B$ is an one one onto (bijective) mapping, then the mapping $f^{-1} : B \rightarrow A$, which associates to element $b \in B$, to the unique element $a \in A$ such that $f(a) = b$ is called the inverse of the mapping $f : A \rightarrow B$.

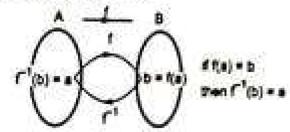


Fig. 2.20

Condition for a function to be invertible :

1) If $f : A \rightarrow B$ is one one onto mapping then only f^{-1} exists and f^{-1} is also one one onto mapping.

2) In case of one one into, many one into and many one onto, inverse does not exist as in case of,

i) For $f^{-1} : B \rightarrow A$, $p \in B$ is not corresponding with any element of A, hence f^{-1} does not exist.

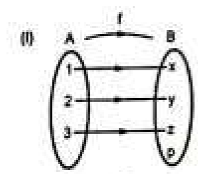


Fig. 2.21

ii) For $g^{-1} : Z \rightarrow Y$, $z_3 \in Z$ is not corresponding with any element of Y and z_1 is having two different images.

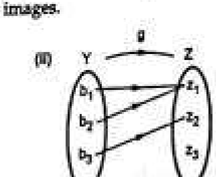


Fig. 2.22

hence g^{-1} does not exist.

iii) For $h^{-1} : B_1 \rightarrow A_1, b_3 \in B_1$ is having two different images, hence h^{-1} does not exist.

Hence f^{-1} exists only in case of f is one one and onto mapping.

(iii)

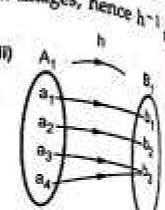


Fig. 2.23

Theorem :

If $f : A \rightarrow B$ is one one onto (bijective) mapping, show that inverse of f, i.e. $f^{-1} : B \rightarrow A$ is also one one onto mapping.

Proof :

f is one one onto hence

i) $\forall a_1, a_2 \in A$

$$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$$

ii) $\forall b \in B \exists a \in A$

$$\text{s.t. } b = f(a)$$

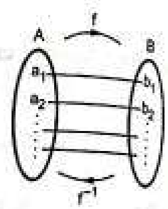


Fig. 2.24

To prove that f^{-1} is one one onto.

Let $f(a_1) = b_1$ and $f(a_2) = b_2$

$\Rightarrow f^{-1}(b_1) = a_1$ and $f^{-1}(b_2) = a_2$

Now $f^{-1}(b_1) = f^{-1}(b_2)$

$\Rightarrow a_1 = a_2$

$\Rightarrow f(a_1) = f(a_2)$ [as f is well defined]

$\Rightarrow b_1 = b_2$

$\Rightarrow f^{-1}$ is one one function.

Again $\forall b \in B \exists a \in A$, s.t $f(a) = b$

$\Rightarrow f^{-1}(b) = a$

hence $\forall a \in A \; \exists b \in B$

s.t. $f^{-1}(b) = a$

hence f^{-1} is onto mapping.

Therfore f^{-1} is one one onto mapping from B to A.

2.8 Composition of Functions or Product of Functions

Let A, B and C be three sets and let f be a mapping from A to B.

i.e $f : A \rightarrow B$. And let g be a mppaing from B to C

i.e. $g : B \rightarrow C$.

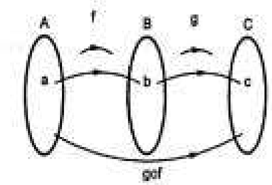


Fig. 2.25

then gof, composite of f and g is a function from A to C.

$$\text{gof}(a) = g[f(a)] = g[b] = c, \; \forall a \in A$$

Domain of gof is set A and co-domain of gof is set B.

Example 2.2 *Three functions F, G, H are defined from $\Re \rightarrow \Re$ as follows :*

i) $f(x) = 2x^3 + 5$

ii) $g(x) = \cos x$

iii) $h(x) = x^3 - 1$

Find ho(gof) and (hog)of. Are they equal ?

Solution : ho(gof) (x)

$$= h[(gof)(x)]$$
$$= h[g[f(x)]]$$
$$= h[g(2x^3 + 5)]$$
$$= h[\cos(2x^3 + 5)]$$
$$= \cos^3(2x^3 + 5) - 1$$

also [hog]of (x)

$$= (hog)[(2x^3 + 5)]$$
$$= h[g(2x^3 + 5)]$$
$$= h[\cos(2x^3 + 5)]$$

$= \cos^3(2x^3+5)-1$

yes, ho(gof) and (hog)of are equal i.e. ho(gof) = (hog)of product of functions satisfy assocoative property.

Example 2.3 *Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$ repectively. Find :*

i) gof (4) and fog (4)

ii) gof (a + 2)

iii) fog (a + 2)

Solution : i) gof(4)

$= g[(f(4)]$

$= g[24 + 1]$

$= g[9]$

$= (9)^2 - 2$

$= 81 - 2$

$= 79$

fog(4)

$= f[g(4)]$

$= f[(4)^2 - 2]$

$= f[14]$

$= 2(14) + 1$

$= 29$

ii) gof(a + 2)

$= g[f(a + 2)]$

$= g[2(a + 2) + 1]$

$= g[2a + 5]$

$= (2a+5)^2 - 2$

$= 4a^2 + 20a + 23$

iii) fog(a + 2)

$= f[g(a + 2)]$

$= f[(a+2)^2 - 2]$

$= f[a^2 + 4a + 2]$

$= 2[a^2 + 4a + 2] + 1$

$= 2a^2 + 8a + 5$

Example 2.4 *Let functions f and g be defined by $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, Find*

1) gof (a + 3)

2) fog (a + 3)

3) fog (5)

Solution :

1) gof(a + 3) $= g[f(a + 3)]$

$= g[2(a + 3) + 1]$

$= g(2a + 7)$

$= (2a+7)^2 - 2$

$= 4a^2 + 28a + 47$

2) fog(a + 3) $= f[g(a + 3)]$

$= f[(a+3)^2 - 2]$

$= f(a^2 + 6a + 7)$

$= 2(a^2 + 6a + 7) + 1$

$= 2a^2 + 12a + 15$

3) fog(5) $= f[g(5)]$

$= f[(5)^2 - 2]$

$= f(25 - 2)$

$= f(23)$

$= 2(23) + 1$

$= 47$

Example 2.5 *Represent the given function in : i) Graphical ii) Tabular and iii) Matrix form.*

F = {(1, 3), (2, 5), (3, 5), (4, 2), (5, 3)}

Solution : i) Graphical representation

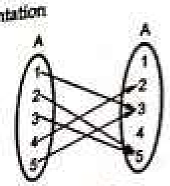


Fig. 2.26

ii) Tabular representation

f	1	2	3	4	5
1			✓		
2					✓
3					✓
4		✓			
5			✓		

iii) Matrix form

$$\begin{array}{c} \\ 1\\2\\3\\4\\5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Example 2.6 *Let $f(x) = x + 2$, $g(x) = x - 2$, $h(x) = 3x$ for $x \in \Re$, where $\Re$ is the set of real numbers.*

Find : (i) gof, (ii) fog, (iii) fof, (iv) hog,
v) gog , vi) foh, vii) hof, viii) fohog

Solution :

i) $\quad gof\,(x) = g\,[f(x)] = g\,[x + 2]$
$= (x + 2) - 2$
$= x$

ii) $\quad fog\,(x) = f\,[g(x)]$
$= f\,(x - 2)$

$= (x - 2) + 2$
$= x$

iii) $\quad fof\,(x) = f\,[f(x)]$
$= f\,(x + 2)$
$= (x + 2) + 2$
$= x + 4$

iv) $\quad hog\,(x) = h\,[g(x)]$
$= h\,[x - 2]$
$= 3\,(x - 2)$
$= 3x - 6$

(v) $\quad gog\,(x) = g\,[g(x)]$
$= g\,(x - 2)$
$= (x - 2) - 2$
$= x - 4$

(vi) $\quad foh\,(x) = f\,[h(x)]$
$= f\,(3x)$
$= 3x + 2$

(vii) $\quad hof\,(x) = h[f(x)]$
$= h\,[x + 2]$
$= 3\,[x + 2]$
$= 3x + 6$

(viii) $\quad fohog\,(x) = foh\,[g(x)]$
$= foh\,[x - 2]$
$= f\,[h\,(x - 2)]$
$= f\,(3\,(x - 2))$
$= f\,(3x - 6)$
$= 3x - 6 + 2$
$= 3x - 4$

Example 2.7 *Let* $f(x) = x + 5$, $g(x) = x^2$ $x \in \Re$ *then find* $gof(x)$ *and* $fog(x)$

Solution :

$$\begin{aligned} gof(x) &= g[f(x)] \\ &= g[x+5] \\ &= (x+5)^2 \\ &= x^2 + 10x + 25 \end{aligned}$$

also
$$\begin{aligned} fog(x) &= f[g(x)] \\ &= f(x^2) \\ &= x^2 + 5 \end{aligned}$$

$fog \neq gof$.

Remark - Composite of functions satisfy associative property but does not satisfy commutative property.

2.8.1 Some Properties of Composite of Mappings

Theorem 1

If $f : A \rightarrow B$ be a one one and onto mapping, then

$$fof^{-1} = I_B \text{ and } f^{-1}of = I_A$$

where I_A and I_B are the identity mappings on the sets A and B respectively.

Proof :

$f : A \rightarrow B$

and f is one one onto mapping

$\Rightarrow f^{-1}$ exists

and $f^{-1} : B \rightarrow A$ is one one onto mapping.

Now $fof^{-1} : B \rightarrow B$

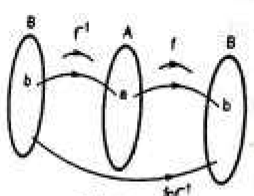


Fig. 2.27

fof^{-1} is a mapping from B to B

$$\begin{aligned} fof^{-1}[b] &= f[f^{-1}(b)] \\ &= f(a) \\ &= b \quad \forall\, b \in B \end{aligned}$$

hence $fof^{-1} = I_B$

also $f^{-1}of : A \rightarrow A$

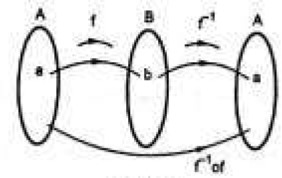


Fig. 2.28

$$\begin{aligned} & f^{-1} of (a) \\ &= f^{-1}[f(a)] \\ &= f^{-1}(b) \\ &= a \end{aligned}$$

Hence $f^{-1} of = I_A$

Theorem 2

If $f : A \rightarrow B$ and $g : B \rightarrow C$ be two one one onto function then $gof : A \rightarrow C$ is also one one onto and

$$(gof)^{-1} = (f^{-1} og^{-1})$$

Proof :

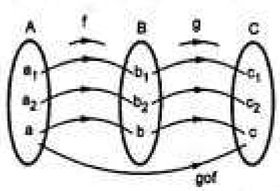


Fig. 2.29

gof : A → C

To show that gof is one one function

Let $gof(a_1) = gof(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow g(b_1) = g(b_2)$

$\Rightarrow b_1 = b_2$ [as g is one one function]

$\Rightarrow f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$ [as f is also one one function]

Hence gof is one one function. To show that gof is onto function. Let c be arbitrary element of C.

Since g is onto mapping from B to C. Therefore ∃ b ← B. s. t.

$g(b) = c$

Again since f is an onto function from A to B therefore ∃ a ∈ A, such that

$f(a) = b$

Thus ∀ c ∈ C ∃ a ∈ A

s.t. $c = gof(a)$

Hence gof is onto function. Therefore gof is one one onto function and hence gof invertible.

$(gof)^{-1}$ is a mapping from C to A also $(gof)^{-1}$ is one one onto mapping.

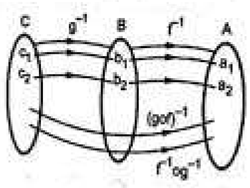


Fig. 2.30

$f^{-1}og^{-1}$ and $(gof)^{-1}$ are the two functions from C to A to show that $(gof)^{-1} = f^{-1}og^{-1}$ we have to show that ∀ c ∈ C

$(gof)^{-1}c = (f^{-1}og^{-1})(c)$

$(g\,of)\,a = c$

$\Rightarrow \quad (g\,of)^{-1}c = a$

also $(f^{-1}og^{-1})\,c = f^{-1}[g^{-1}(c)]$

$= f^{-1}(b)$

$= a$

Hence $\boxed{(g\,of)^{-1} = f^{-1}og^{-1}}$

Theorem 3

Composite of function satisfies associative property.

Let f : A → B, g : B → C and h : C → D, then (hog)of = ho(gof).

Proof :

(hog)of and ho(gof) both mappings are defined from A to D.

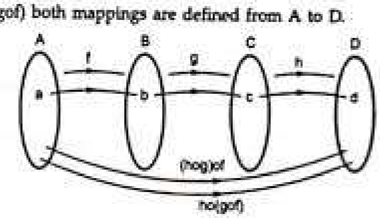


Fig. 2.31

$(hog)of\,(a) = hog\,[f(a)]$

$= hog\,(b)$

$= h\,[g(b)]$

$= h(c)$

$= d$

$ho(gof)\,(a) = h\,[gof\,(a)]$

$= h\,[g\,(f\,(a))]$

$= h\,[g\,(b)]$

$= h\,(c)$

$= d$

Hence ∀ a ∈ A(hog)of (a) = ho(gof) (a) hence (hog)of = ho(gof).

2.9 Comparability of Sets

Comparable : If A and B are two sets and if $A \subset B$ or $B \subset A$, then set A and B are said to be comparable.

Incomparable : If A is not subset of B and B is not subset of A then A and B are said to be incomparable.

2.9.1 One One Correspondence

Let A and B be two sets. If $\exists$ a mapping f from A to B such that f is one one onto mapping then it is also known as A and B are in one one correspondence and is denoted by $A \leftrightarrow B$.

If A and B are in one one correspondence then $n(A) = n(B)$ [i.e. number of elements in A are same as number of elements in B]

i.e. $|A| = |B|$

2.10 Cardinally Equivalent Sets

Two sets A and B whose numbers can be placed in one one correspondence are said cardinally equivalent or equinumerous.

In other words if $\exists$ a mapping $f : A \to B$ which is one one onto (bijective) then A and B are equivalent. They are denoted by $A \sim B$.

Example

$$A = \{1, 2, 3, 4\}, \quad B = \{p, q, r, s\}$$

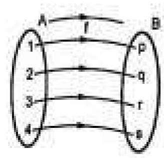


Fig. 2.32

$f : A \to B$ is bijective, hence $A \sim B$.

i.e. $|A| = |B| = 4$

From set A to set B

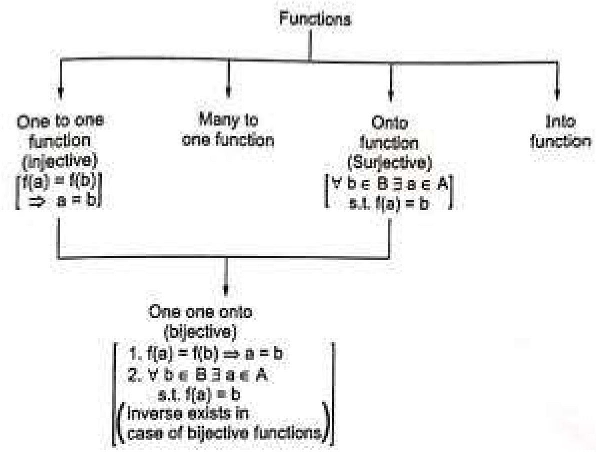


Fig. 2.33

3 Counting

Syllabus

The Basics of Counting, The Pigeonhole Principle, Permutations and Combinations, Binomial Coefficients, Generalized Permutations and Combinations, Generating Permutations and Combinations.

Contents

3.1 Countable Sets

3.1.1 Denumerable Set

An infinite set is said to be denumerable if its elements can be put in one one correspondence with the set of natural numbers.

i.e. $\exists$ at least one mapping f from set A to set N. Such that f is one one and onto, then set A is know as denumerable.

If such a correspondence does not exist then the set is known as non-denumerable.

3.1.2 Countable Sets

A set which is either empty or finite or denumerable is called **countable** set.

Uncountable Sets

A set which is non-denumerable is called as uncountable set.

Examples

1. $\emptyset$ is countable set and $|\emptyset| = 0$
2. $A = \{a, b, c, d\}$ is countable set and $|A| = 4$.
3. The set of all integers is countable.

Proof of (3)

A one one correspondence between I and N can be made as follows.

$$\begin{array}{ccccccccc} I: & 0 & 1 & -1 & 2 & -2 & \cdots m & & -m \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\ N: & 1 & 2 & 3 & 4 & 5 & 2m & & 2|m|+1 \end{array}$$

Fig. 3.1

i.e. f is a function from I to N such that

$$f(m) = 2\,m \text{ if m is positive}$$

and $$f(m) = 2|m|+1 \text{ if m is negative or zero.}$$

Example 3.1 *Prove that the set of rational numbers in [0, 1] is countable.*

Solution : To show that set of rational numbers in [0, 1] is countable, we have to show that $\exists$ at least one function f from [0, 1] to N. Such that f is one one onto.

We arrange the rational numbers of the interval according to increasing denominators as $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \cdots$

then the one one correspondence can be indicated as,

$$1 \leftrightarrow 0$$
$$2 \leftrightarrow 1$$
$$3 \leftrightarrow \frac{1}{2}$$
$$4 \leftrightarrow \frac{1}{3}$$
$$5 \leftrightarrow \frac{2}{3}$$
$$6 \leftrightarrow \frac{1}{4}$$
$$7 \leftrightarrow \frac{3}{4}$$
$$8 \leftrightarrow \frac{1}{5} \text{ etc.}$$

Hence the set is countable.

Example 3.2 *Prove that the set of real numbers in [0, 1] is not countable.*

Solution : Any real number between [0, 1] can be written in the following manner in a unique decimal form.

$a_1, a_2\ a_3 \ldots\ldots a_i \ldots\ldots$

Where a_i is any of the digits 0, 1, 2, ... 9. Assume that the set A of all real numbers in [0, 1] is countable. Then the elements of A can be written in a sequence. Therefore

$$A = \{x_1, x_2, \ldots x_n, \ldots\}\ x_i \in A$$

Now each $x_i \in A$ can be written in decimal form as

$$x_1 = 0 \cdot a_{11}\ a_{12}\ a_{13} \cdots$$
$$x_2 = 0 \cdot a_{21}\ a_{22}\ a_{23} \cdots$$
$$x_3 = 0 \cdot a_{31}\ a_{32}\ a_{33} \cdots$$

....

....

$$x_n = 0 \cdot a_{n1}\, a_{n2}\, a_{n3} \cdots$$

....

Since all the elements of A are countable. We can establish one one correspondence between the elements of A and natural numbers.

$$1 \leftrightarrow 0 \cdot a_{11}\, a_{12} \cdots$$

$$2 \leftrightarrow 0 \cdot a_{21}\, a_{22} \cdots$$

$$3 \leftrightarrow 0 \cdot a_{31}\, a_{32} \cdots$$

Now construct a decimal number

$$y = 0 \cdot b_1\, b_2\, b_3 \cdots$$

$$b_i = 0 \text{ if } a_{ii} \neq 0$$

$$= 1 \text{ if } a_{ii} = 0$$

Hence $b_1 \neq a_{11},\ b_2 \neq a_{22},\ b_3 \neq a_{33} \cdots$

Hence $b_i \neq a_{ii}$ for any i.

Hence $y \neq x_1$ since it differs from x_1 at the first decimal place.

$y \neq x_2$ since it differs from x_2 at the second decimal place

and in general

$y \neq x_i$ Since it differs x_i and i^{th} place.

Thus the number $y = 0 \cdot b_1\, b_2$ ...... lies between 0 and 1 and is different from each of the numbers in the above list and therefore is not an element of A.

This contradicts our assumption that A is countable.

Hence A is not countable.

3.1.3 Properties of Countable Sets and Uncountable Sets

1. A subset of a countable set is countable.
2. A super set of an uncountable set is uncountable.
3. Union of countable sets is countable i.e. if A and B are countable sets then $A \cup B$ is also countable.
4. Cartesian product of countable sets is countable.
   i.e. if A and B are countable sets then $A \times B$ is also countable.
5. Union of countable collection of countable set is countable.
   i.e. $\bigcup_{i=1}^{n} A_i$ is countable if each A_i is countable.

Remarks :

1. Set of real numbers in [0, 1] is an uncountable set hence super set of [0, 1], i.e. real numbers set is also uncountable set.
2. Set of irrational numbers is uncountable.
3. P (N), [Power set of natural numbers] is uncountable set.

3.2 Pigeon Hole Principle

Pigeon hole principle says that if there are 'many' pigeons and 'a few' pigeon holes, then there must be some pigeon holes occupied by two or more pigeons.

Let A and B be finite sets and $|A| > |B|$ then for any function f from A to B, $\exists\, a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$.

Example 1

If 11 shoes are selected from 10 pairs of shoes there must be a pair of matched shoes among the selection.

Here 11 shoes are the pigeons and the 10 pairs are the pigeon holes.

[**Remark** : Pigeon hole principle is also known as the shoe box argument].

3.2.1 The Extended Pigeon Hole Principle

If there are m pigeon holes and more than 3 m pigeons then 4 or more pigeons will occupy at least one of the pigeon hole. In general if the number of pigeons is much larger than the number of pigeon holes, the pigeon hole principle can be extended as,

Theorem

If n pigeons are assign to m pigeon holes, then one of the pigeon holes must be occupied by, at least $\left[\frac{n-1}{m}\right]+1$ pigeons.

Remark : $\left[\frac{n-1}{m}\right]$ is integer division of n-1 by m e.g. $\left[\frac{10}{3}\right] = 3$ or $\left[\frac{16}{3}\right] = 5$, $\left[\frac{9}{2}\right] = 4$ ...

Example 2

Show that any 30 people are selected, then at least 5 people must have been born on the same day of the week.

By extended pigeon hole principle.

$n = 30$, $m = 7$. At least $\left[\frac{30-1}{7}\right]+1$ are born on the same day i.e. $4 + 1 = 5$ people have been born on the same day.

Example 3.3 *Show that 7 colours are used to paint 50 bicycles, at least 8 bicycles will be of same colour.*

Solution : By extended pigeon hole principle at least $\left[\frac{n-1}{m}\right]+1$ pigeons will occupy one pigeon hole.

Hence $n = 50$, $m = 7$

then $7 < 50$

$$\left[\frac{50-1}{7}\right]+1 = 7+1$$

$= 8$ bicycles will be of same colour.

Example 3.4 *Show that if seven distinct numbers from 1 to 12 are chosen then two of them will add upto 13.*

Solution :

$A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$ $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$ are the six different set containing two distinct numbers such that the sum of two numbers is 13. Now there are only 6 such sets, hence by pigeon hole principle two of the chosen numbers must belong to the same set.

Permutation and Combination

3.3 Introduction to Permutation and Combination

Here we shall discuss some very useful counting techniques.

For example,

Three students are selected from 4 brilliant students, A, B, C and D.

We can select them in the following possible ways.

ABC, ACD, ABD, BCD. i.e. 4 different selections are possible.

Now rank I, II and III are given to every group of selected students.

Let the first group be A, B, C.

The ranks I, II and III can be given in following ways.

A B C, A C B, B A C, B C A, C A B, C B A. i.e. 6 different ways.

Similarly,

ABD, ACD and BCD will have 6 different ways for giving ranks I, II and III.

The total number of such ways

$$= 4 \times 6 = 24$$

In the first case, the order of the selection is not considered, while in the second case it is important.

In the first case, we select a group and in second case we arrange the elements in order.

The selection of object without consideration of the order of their selection is called **combination.**

The selection of object with consideration of the order is called the **permutation.** i.e. **Permutation** is an arrangement of objects and combination is a selection of objects.

I) Rule of Product

If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m \times n$ possible outcomes when both of these experiments take place.

II) Rule of Sum

If one experiment has m possible outcomes and another experiment has n possible outcomes, then there are $m + n$ possible outcomes when exactly one of these experiments takes place.

[Remark - In counting the word 'and' is replaced by multiplication, '·' and the word 'or' is replaced by addition, '+'.]

3.4 Permutations

Consider a problem, that we have 5 students and 20 chairs. Students wish to sit on these chairs.

1^{st} student can have 20 choices to sit on the chair. Now one chair is occupied.

The 2^{nd} students can sit on any one of the remaining 19 chairs. i.e. 2^{nd} students may have 19 choices. This way 3^{rd} student have 18 choices, 4^{th} student have 17 choices and 5^{th} student have 16 choices.

Now 1^{st}, 2^{nd}, 3^{rd}, 4^{th} and 5^{th} (i.e. all 5 students are sitting) the total number of distinct ways to sit on chairs

$$= 20 \times 19 \times 18 \times 17 \times 16 \qquad \text{... (By the rule of product)}$$

Definition

An arrangement in sequence of elements of a set is called a permutation of the elements.

Essentially, there are three types of arrangement of elements to be considered.

Type I

Let $0 \leq r \leq n$. The number of ways to have an ordered sequence of n distinct elements, taken r at a time is called as an r-permutation of n elements and is denoted by P(n, r) or ${}^nP_r$.

As discussed in introduction, the first place in the sequence can be filled up in n-ways, the second place in (n − 1) ways and proceeding in this manner the r^{th} place can be filled up in (n − r + 1) ways.

Hence $P(n, r)$ or ${}^nP_r = n(n-1)(n-2)\ldots(n-r+1)$

or
$$\boxed{{}^nP_r = \frac{n!}{(n-r)!}}\ , 0 \leq r \leq n$$

Example 3.5 *Compute number of permutations of the set given {1, 2, 3, 4, 5}.*

Solution : The number of permutations of the set {1, 2, 3, 4, 5} are 5!.

Example 3.6 *Find the number of permutations of A taken r at a time.*
A = {a, b, c, d, e, f}, r = 2

Solution : The number of permutations of A taken r at a time = ${}^6P_2$

$$= \frac{6!}{4!} = 30$$

Example 3.7 *How many four digit numbers can be formed out of the digits 1, 2, 3, 4, 5, 7, 8, 9, if i) No repetition is permitted.*
ii) How many of these will be greater than 3000.

Solution : i) Out of 8 digit, 4 digit number can be formed in $8 \times 7 \times 6 \times 5$ ways.

$$= 1680$$

ii) There is a restriction that the 4 digit numbers so formed must be greater than 3000.

Thousand Place	Hundred	Ten	Unit

Thousand's position can be filled out of the numbers 3, 4, 5, 7, 8, 9.

i.e. thousand's position can be selected in 6 different ways.

Now out of remaining 7 digits, Hundred's position can be filled in 7 different ways. Ten's position can be filled in 6 different ways and unit's position can be filled in 5 different ways.

Hence total number of 4 digit number, greater than 3000 can be formed in $6 \times 7 \times 6 \times 5$ different ways

$$= 1260 \text{ ways}$$

Example 3.8 *i) Suppose repetitions are not permittable, how many four digit numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ?*
ii) How many of such numbers are less than 4000 ?
iii) How many in (i) are even ?
iv) How many in (ii) are odd ?
v) How many in (i) contain both 3 and 5.
vi) How many in (i) are divisible by 10.

Solution : i) Out of 6 numbers, 4 digit numbers can be formed in ${}^6P_4$ ways.

$$\therefore \text{Number of ways} = \frac{6!}{2!} = 360$$

ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3. i.e. 1^{st} digit can be chosen in 3 ways, 2^{nd} digit can be any one of the remaining 5 digits. 3^{rd} digit can be any of the remaining 4 digit and the 4^{th} digit is any one of the remaining 3 digits. Hence the total number of ways

$$= 3 \times 5 \times 4 \times 3 = 180$$

iii) Those numbers ending in 2 or 8 are even numbers.

Hence the last digit (4^{th} digit) can be chosen in 2 ways (the number 2 or 8). The first digit can be chosen in any one of the remaining 5 digits, 2^{nd} in any of the 4 digits and 3^{rd} in any of the 3 digits. Hence the total number of ways

$= (5 \times 4 \times 3) \times 2$

Thousand place	Hundred	Ten	Unit

= 120 ways

iv) The numbers less than 4000 and are odd.

The numbers ending with 1, 3, 5 or 7 are odd.

The 4 digit numbers ending in 1 and less than 4000 should begin with either 2 or 3.

Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similary the number ending in 3 are 24. However the number ending with 5 or 7 are

Thousand's place, Unit place

$3 \times 4 \times 3 \times 2$

= 72 ways

Hence the total number which are odd and less than 4000 are

$= 24 + 24 + 72$

$= 120$

v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be 4×3, i.e. 12.

Now the remaining two positions will be filled by the remaining 4 numbers i.e. 1, 2, 7 and 8.

Hence out of remaining two positions one positon can be occupied in 4 different ways and the remaining position will be occupied in 3 different ways.

Hence total number of 4 digit numbers in which both 3 and 5 are present

$= 12 \times 4 \times 3$

$= 144$

vi) Not even a single number is divisible by 10 as there is no zero at unit's place.

Example 3.9 *Out of nine cabins in office, in how many ways four cabins can be assigned to four officers.*

Solution : The first officer may choose cabin in 9 ways.

Second officer may choose the cabin in 8 ways

Third officer may choose the cabin in 7 ways and the fourth officer in 6 ways.

Hence the total number of distinct ways

$= 9 \times 8 \times 7 \times 6$

Example 3.10 *A menu card in a restaurant displays four soups, five main courses, three desserts and 5 beverages. How many different menus can a customer select if :*

i) He selects one item from each group without omission.

ii) He chooses to omit the beverages, but selects one each from the other groups.

iii) He chooses to omit the desserts but decides to take a beverage and one item each from the remaining groups.

Solution : i) The customer can select the soup in 4 ways, the main course in 5 ways, the dessert in 3 ways and beverages in 5 ways.

Hence by the product rule, the number of ways in which he can select one item each, without omission is $4 \times 5 \times 3 \times 5 = 300$.

ii) The number of ways is which he omit beverages,

$= 4 \times 5 \times 3$

$= 60$ ways

iii) The number of ways in which he omit desserts but he takes all other items,

$= 4 \times 5 \times 5$

$= 100$ ways

Example 3.11 *Suppose that repetitions are not permitted, how many five digits decimal numbers can be formed.*

Solution : Out of five digits, the first digit can be any of nine digits 1, 2, ...9 (as 0 cannot be the first digit), the second digit can be any of the nine remaining digits.

Third digit can be any of the 8 remaining.

Fourth can be any of the 7 remaining and the last digit can be any of the 6 remaining.

Hence the total number of five digits can be formed in $9 \times 9 \times 8 \times 7 \times 6$ ways.

Example 3.12 *In how many ways can three examination be scheduled within a five days period so that no two examinations are scheduled on the same day ?*

Solution : The first examination can be scheduled on any of the 5 days.

Hence there are 5 ways to schedule first examination.

Second examination can be scheduled in 4 ways and third examination can be scheduled in 3 ways.

Hence total number of ways $= 5 \times 4 \times 3$

$= 60$ ways

Example 3.13 *2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total number of ways if.*

i) Mathematics papers are consecutive. ii) Mathematics papers are not consecutive.

Solution : i) Both mathematics papers (M_1 and M_2) are together.

Consider both M_1 and M_2 as single paper.

These two papers among themselves can be arranged in 2! ways.

Now 6 papers (as M_1 and M_2 is considered as single paper) can be arranged in 6! ways.

Hence total number of arrangements

$= 2! \cdot 6!$

ii) If M_1 and M_2 are not consecutive then they are to be arranged between the 4 gaps or at the 2 ends.

M_1 or M_2 ▭ M_1 or M_2 ▭ M_1 or M_2 ▭ M_1 or M_2 ▭ M_1 or M_2 ▭ M_1 or M_2

where ▭ denotes other papers.

Hence there are 6 places where mathematics papers can be arranged.

Therefore, 2 mathematics papers can be arranged in 6 places in ${}^6P_2$ ways.

Five other papers can be arranged among themselves in 5! ways.

Therefore total number of arrangements

$= 5! \cdot {}^6P_2$

$= 5! \cdot 6 \cdot 5$

$= (120) \cdot (30)$

$= 3600$

Example 3.14 *10 different M_1 books, 3 different M_2 books, 5 different M_3 books and 7 different D.S. books are to be arranged on a shelf. How many different arrangements are possible if*

i) The books in each subjects must all be together ?

ii) Only M_3 books must be together.

Solution :

i) M_1 books can be arranged among themselves in 10! ways, the M_2 books in 3! ways, M_3 books in 5! ways and D.S. books in 7! ways.

Hence the total number of arrangements

$= 4! \cdot 10! \cdot 3! \cdot 5! \cdot 7!$

ii) Consider the 5 M_3 books as a single book. Then there are 21 books which can be arranged in 21! ways. In each of these arrangements, the M_3 books can be arranged among themselves in 5! ways.

Hence, the number of arrangements in $5! \cdot 21!$

Example 3.15 *In how many ways letter 'ORGANISE' can be arranged in such a way that all vowels always come together.*

Solution : The word 'ORGANISE' has 8 letters, of these 4 are vowels and 4 are consonants.

In the arrangement of these 8 letters, the restriction is that the 4 vowels should be together. Thus, we can take the 4 vowels as one object. The 4 vowels O, A, I, E in this one object can be arranged in 4! ways.

Now we have 5 letters, i.e. 4 consonants and one compound letter consisting of 4 vowels. The total number of ways of arranging them is ${}^5P_5 = 5!$.

Hence, the total number of arrangements of the letters of the word 'ORGANISE', in which all the vowel letters come together

$= 2! \cdot 5! \cdot 4!$

Type II

We derive now a general formula for the number of ways to place r coloured balls in n number boxes, where m_1 of these are of one colour, m_2 of them are of a second colour ..... and m_r of them are of a r^{th} colour.

Here the placement of the r balls is not changed by rearranging the m_1 balls of the same colour among the boxes in which they are placed or rearranging the m_2 balls of the same colour among the boxes in which they are placed, —

On the other hand, if the r balls were distinctly coloured, any arrangement will y[e]
a different placement.

It follows that each way to place the r not completely distinctly coloured b[alls]
corresponds to $m_1!\ m_2! \ldots m_r!$ ways to place r distinctly coloured balls. Since there [are]
P(n, r) ways to place r distinctly coloured balls in n numbered boxes, the total number [of]
ways to place r coloured balls in n numbered boxes, where

m_1 of these balls are of one colour,

m_2 of these balls are of a second colour,

⋮

m_r of these balls are of a r^{th} colour, is $\dfrac{P(n, r)}{m_1!\ m_2! \ldots m_r!}$

i.e. $$\boxed{\text{Number of ways} = \dfrac{{}^nP_r}{m_1!\ m_2! \ldots m_r!}}$$

Example 3.16 *Find the number of ways to paint 15 offices so that 4 of them will be blue, 3 of them green, 5 of them yellow and remaining all white.*

Solution : Total number of permutations

$$= \frac{15!}{4!\ 3!\ 5!\ 3!}$$

Example 3.17 *Find the number of permutations obtained by arranging all letters of the word 'COMBINATION'.*

Solution : Out of 11 letters, O, I and N are the letters repeated 2 times.

Hence total number of permutations are

$$\frac{11!}{2!\ 2!\ 2!} = \frac{11!}{8}.$$

Example 3.18 *Find the number of distinct permutations that can be formed from all the letters of each word i) RADAR ii) UNUSUAL*

Solution : i) Total number of letters in RADAR are 5, in which A is repeated twice.

Hence total number of distinct arrangements

$$= \frac{5!}{2!} = 60$$

ii) Total number of letters in UNUSUAL are 7, in which U is repeated 3 times.

Hence total number of distinct arrangements

$$= \frac{7!}{3!} = 840$$

Example 3.19 *Determine the number of ways in which letters in the word PIONEER be arranged. So that two E's are always together.*

Solution : The other five letters in the word can be arranged in 5! ways and for each such arrangement the two E's can occupy any of the 6 remaining places.

Total number of arrangements

$$= 6 \cdot 5!$$
$$= 6!$$
$$= 720$$

Example 3.20 *How many distinguishable permutations of the letters in the word 'BANANA' are there ?*

Solution : The total number of distinguishable permutations of the letters in the word 'BANANA' are $\dfrac{6!}{3!\ 2!}$

[As letter 'A' occurs 3 times, hence there are 3! ways in which these 'A' can be rearranged among themselves and letter 'N' occurs 2 times, hence there are 2! ways in which these 'N' can be rearranged among themselves. Also number of letters in BANANA are 6].

Type III

The number of permutations of n elements, r at a time, when each element may be repeated once, twice, ... upto r times in any arrangement.

In this case, the first place may be filled up in n ways, the second place may also be filled up in n ways, and so on .... Proceeding in this manner, the number of ways in which the r places can be filled up is n^r.

Example 3.21 *A bit is either 0 or 1. A byte is a sequence of 8 bits. Find*

i) Number of bytes.

ii) Number of bytes that begin with 11 and end with 11.

Solution : i) Total number of bytes is 2×2 .... (8 times)

$$= 2^8 = 256$$

ii) Since the first two and last two bits are fixed, i.e. 11, the remaining bits in the sequence are either 0 or 1.

Hence the total number of bytes

$= 2 \times 2 \times 2 \times 2$

$= 2^4 = 16$

Example 3.22 *How many auto license plates can be made if each identified by 2 letters followed by 4 digits.*

Solution : The first two positions in the sequence can be occupied in $26 \times 26 = 26^2$ ways. The next 4 positions can be occupied in

$= 10 \times 10 \times 10 \times 10$

$= 10^4$ ways

Hence the number of auto license plates that can be made is

$= 26^2 \times 10^4$

Example 3.23 *A die is rolled four times, find the number of faces that can appears on top.*

Solution : If the die is rolled once, the faces appearing on the top can be any one of the six faces 1 to 6.

When it is rolled second time, again there are 6 choices for the face appearing on top and same is the situation when the die is rolled third or fourth times.

Hence the total number of ways of a face appearing on top is

$= 6 \times 6 \times 6 \times 6$

$= 6^4$

Example 3.24 *There are 9 switches on a fuse box. How many different arrangements are there ?*

Solution : Each switch has 2 possible positions on or off. Placing a 2 in each of the 9 positions we have

$2^9 = 512$ ways

Example 3.25 *How many 3 letter words can be created, if repetitions are allowed ?*

Solution : There are 26 letters to choose from, and we are allowed to have repetitions.

There are $26 \cdot 26 \cdot 26 = 26^3$ possible 3 letter words.

Example 3.26 *In how many ways can 10 letters be posted in 4 letter boxes ?*

Solution : The first letter can be posted in 4 ways, the second can be posted in 4 ways and so on.

Hence the number of ways

$= 4 \cdot 4 \cdot 4$ ...... 10 times

$= 4^{10}$

Example 3.27 *There are 10 true/false questions on a test and 5 multiple choice questions with 4 possible answers. How many different choices for answering the 15 questions ?*

Solution : If the student attempts to answer all the questions, 5 multiple choice questions can be answered in 4^5 ways and 10 true/false questions can be answered in 2^{10} ways. Hence the total number of choices for answering the 15 questions

$= 4^5 \times 2^{10}$

Miscellaneous Examples

Example 3.28 *In how many ways five boys and five girls are to be seated in a row if*

i) All the boys must be seated in five leftmost seats ?

ii) No two boys can be seated together ?

iii) John and Mary must be seated together ?

Solution :

i) Since all the boys must be seated in the five leftmost seats, then boys can be seated in 5! ways. Also girls must be seated in the rightmost seats and girls can be seated in 5! ways.

Hence total number of ways in which they can be seated in this way = 5!$\cdot$5!.

ii) First let the girls be seated. This arrangement can be done in 5! ways. Now for each such seating arrangement there will be (5 − 1) = 4 gaps between any adjacent pair of girls, in addition to the 2 gaps at the extreme ends. Hence the seating arrangement of the boys should be such that these 6 gaps have to be filled in 5 ways so that no gap should be assigned to more than one boy. This will be done in P(6, 5) ways. Hence the number of ways of seating the boys, so that no two are seated adjacent is $5! \cdot {}^6P_5$.

$= 5! \cdot 6!$ ways

iii) Let us consider John and Mary as single entity. They can be seated together in the following ways.

(1, 2) (1st and 2nd position), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9) and (9, 10)th positions. i.e. 9 positions and John and Mary among themselves can be seated in 2! ways.

Hence John and Mary can be together in $2! \times 9$ ways.

Now the remaining 8 people can be seated in 8! ways.

Hence the total number of ways in which the boys and girls are seated so that John and Mary are seated together

$= 8! \cdot (18)$

Example 3.29 *Find the number of ways in which five persons*

i) Can sit in a row

ii) How many ways are there if two of the persons insists on sitting next to one another.

Solution : i) Number of ways in which 5 persons can sit in a row

$= 5 \times 4 \times 3 \times 2 \times 1$

$= 5!$

ii) Consider two persons as single entity then there are total 4 persons. These persons can sit in 4! different ways.

And two persons which we have assumed as single entity can sit in 2! ways. Hence total number of arrangements

$= 4! \times 2!$

$= 24 \times 2$

$= 48$

Example 3.30 *The number plates in Pune start with the initials PU and are of the form PU NA, NNNN where N represents a number with the condition that the first N is not equal to zero. Also, the last 4 N's cannot be all equal to zero and A represents an alphabet. How many maximum number plates can be issued in Pune ?*

Solution : The first N can have the value 1 to 9. The alphabet A can be selected from 26 alphabets. The last 4 Ns can be filled in $10 \times 10 \times 10 \times 10 - 1$ ways (the case of all N equal to 0 is removed).

Therefore, maximum number of number plates that can be issued

$= 9 \times 26 \times 9999$

$= 2339766$

Example 3.31 *Four friends go to a city in which there are 10 hotels.*

i) In how many ways can they stay ?

ii) In how many ways can they stay if no two friends stay together ?

iii) In how many ways can they stay if atleast two friends stay together ?

Solution : i) The first friend can stay in 10 ways, the second friend can stay in 10 ways and so on.

Hence number of ways $= 10 \cdot 10 \cdot 10 \cdot 10$

$= 10^4$

ii) If no two friends can stay together, the first friend can stay in 10 ways, the second friend can stay in 9 ways,

the third friend can stay in 8 ways, the fourth friend can stay in 7 ways.

Hence total number of ways $= 10 \cdot 9 \cdot 8 \cdot 7$

$= 5040$

iii) The total number of ways in which 4 friends can stay together $= 10^4$ and the number of ways in which no two friends stay together $= 5040$.

Hence the number of ways in which atleast any two friends can stay together

$= 10^4 - 5040 = 4960$ ways.

Example 3.32 *i) How many three digit numbers are there ?*

ii) How many three digit numbers are there which have atleast one 7 in them ?

Solution : i) We have to fill three places with three digits. The first digit cannot be 0. Therefore, we can choose the first digit in 9 ways. We can choose second and third digits in 10 ways each.

Therefore, the total number of ways of choosing 3 digit number $= 9 \times 10 \times 10 = 900$.

ii) To find the number of three digit numbers with atleast one 7 in them, we first find the number of 3 digit numbers with no 7 in them and then subtract this number from the total number of 3 digit numbers.

To find the number of 3 digit numbers with no 7 in them, the first digit cannot be 0 or 7. Therefore, we choose first digit in 8 ways, the second and third digits cannot be 7, hence these can be chosen in 9 ways each.

Hence number of 3 digit numbers without digit $7 = 8 \times 9 \times 9 = 648$

Therefore, total number of 3 digit numbers with atleast one 7 in them

$= 900 - 648$

$= 252$

Example 3.33 *How many 3 digit numbers are there which are even and have no repeated digits ?*

Solution : For number to be even it must end in 0, 2, 4, 6 or 8. There are two cases to consider.

1. **The number ends in 0,** then there are 9 possibilities for the first digit and the 8 possibilities for the second since no digit can be repeated.
   Hence there are $9 \times 8 \times 1 = 72$, three digit numbers that end in 0.
2. **The number does not end in 0,** then there are 4 choices for the last digit (2, 4, 6, 8). When this digit is specified, then there are only 8 possibilities for the first digit. Since the number cannot begin with 0. Finally, there are 8 choices for the second digit and therefore there are $8 \times 8 \times 4 = 256$ numbers that do not end in 0.

Hence the total number of 3 digit numbers which are even (and no repetition)

$= 72 + 256$

$= 326$

Example 3.34 *The number of routes between few cities A, B, C, D, E and F are shown in the Fig. 3.2.*

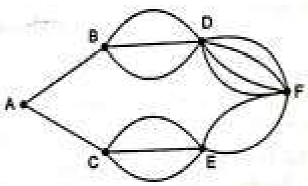


Fig. 3.2

Find : i) In how many ways can you go from city A to city F via city B ?

ii) In how many ways can you go from city A to city F via city C ?

iii) Find the total number of ways you can go from city A to city F.

Solution : i) We can go from A to B in 1 way, from B to D in 3 ways and from D to F in 4 ways. Hence number of ways from A to F via B = $1 \times 3 \times 4 = 12$ ways.

ii) We can go from A to C in 1 way, from C to E in 3 ways and from E to F in 2 ways. Hence number of ways from A to F via C

$= 1 \times 3 \times 2$

$= 6$ ways

iii) Total number of ways from A to F

$= 12 + 6$

$= 18$

Example 3.35 *How many ways can you order the letters of KITCHEN if it must start with a consonant and end with a vowel.*

Solution : Out of 7 letters, there are 5 consonants and 2 vowels.

_ _ _ _ _ _ _

First letter   Last letter

First letter place can be filled in 5 different ways, the last letter place can be filled in 2 different ways, now two places are occupied. Out of remaining 5 letters, remaining 5 places (2^{nd}, 3^{rd}, 4^{th}, 5^{th} and 6^{th}) can be filled in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways.

5 . 5 . 4 . 3 . 2 . 1 . 2

First letter   Last letter

Hence total number of ways $= 5 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot 2$

$= 1200$ ways

Example 3.36 *How many one to one functions are there from a set with 5 elements to one with 7 elements ?*

Solution : Suppose the elements in the domain set are a_1, a_2, a_3, a_4, a_5. There are 7 ways to choose the value of the function at a_1 there are 6 ways to choose value of the function at a_2 and so on.

Therefore, the number of one to one function is

$= 7 \times 6 \times 5 \times 4 \times 3 = 2520$

[**Remark** : There are zero one to one functions from set of 7 elements to set of 5 elements.]

3.5 Combinations

One counting method in which order matters is known as permutation.

The other counting method in which order does not matter is known as combinations.

Definition : $0 \le r \le n$

Selection of a set of r elements from a set of n distinct elements is called a combination.

Consider a problem of choosing 5 mangoes from 12 identical mangoes. We want to know the number of ways in which 5 mangoes can be chosen out of 12 mangoes.

If all the mangoes are identical then according to the rule of permutation, the number of ways $= \dfrac{P(12,\ 5)}{5!}$.

In general the number of ways of choosing r mangoes out of n identical mangoes is

$$= \frac{n(n-1) \ldots\ldots (n-r+1)}{r!}$$

$$= \frac{n!}{r!\,(n-r)!}$$

The quantity $\frac{n!}{r!\,(n-r)!}$ is denoted by $C(n, r)$ or ${}^nC_r$ [out of n choose r]

Hence $\boxed{{}^nC_r = \frac{n!}{r!\,(n-r)!}}$

Relation between ${}^nP_r$ and ${}^nC_r$.

$$\boxed{\frac{{}^nP_r}{r!} = {}^nC_r}$$

or $\boxed{{}^nP_r = r! \cdot ({}^nC_r)}$

Circular arrangements

Let's consider that 4 persons A, B, C and D are sitting around a round table. Shifting A, B, C, D one position in clockwise direction, we get the following arrangements.

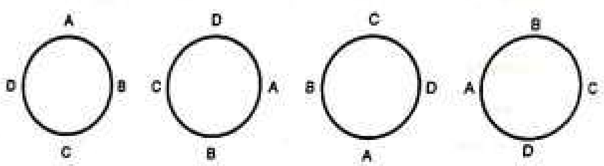


Fig. 3.3

Thus, we use that if 4 persons are sitting at a round table, then they can be shifted four times, but these four arrangements will be the same, because the sequence of A, B, C, D is same. But if A, B, C, D are sitting in a row and they are shifted, then the four linear arrangement will be different.

A-B-C-D, B-C-D-A, C-D-A-B, D-A-B-C we can see that in circular permutation these four arrangements have reduced to one arrangement. Likewise, in circular permutation of n things, a set of n linear arrangements reduce into one arrangement.

Hence the number of permutations in circular arrangement

$$= \frac{\text{Number of permutation in linear arrangement}}{n}$$

$$= \frac{n!}{n} = (n-1)!$$

We can think of circular arrangement in terms of orientation also. The first person can sit in only 1 way since all places are the same. The rest of the persons can now sit with respect to the first person in $(n-1)!$ ways.

When clockwise and anticlockwise arrangements are not different, then observation can be made from both sides and this will be the same. This happens when we have beads in a necklace or flowers in a garland. Here two permutations will be counted as one, so total permutations will be half and in this case,

Circular permutations for pearls in a necklace etc. $= \frac{(n-1)!}{2}$

Example 3.37 *A committee of 12 students consist of 3 representatives from first year, 4 students from second year and 5 from third year. Out of these 12 students, 3 are to be excluded from the committee by drawing lots. What is the chance that :*

i) 3 students belong to 3 different classes.

ii) 2 belong to one class and 1 belong to another class.

iii) Three belong to same class.

Solution : Among 12 students if 3 are excluded,

i) They can be chosen as 1 from first year 1 from second year and 1 from third year as they belong to different classes.

Hence, Chances $= {}^3C_1 \cdot {}^4C_1 \cdot {}^5C_1$

$= 3 \cdot 4 \cdot 5$

$= 60$ ways

ii) 2 students belong to same class and 1 from another class, then the cases are 2 from 1^{st} year and 1 from second year or 2 from 1^{st} year and 1 from 3^{rd} year or 2 from second year and 1 from 1^{st} year or 2 from second year and 1 from third year or 2 from 3^{rd} year and 1 from 1^{st} year or 2 from 3^{rd} year and 1 from 2^{nd} year.

$$= {}^3C_2 \cdot ({}^4C_1 + {}^5C_1) + {}^4C_2 \cdot ({}^3C_1 + {}^5C_1) + {}^5C_2 \cdot ({}^3C_1 + {}^4C_1)$$

$$= 3 \cdot (4+5) + 6 \cdot (3+5) + 10 \cdot (3+4)$$

$$= 60 + 90 + 120$$

$$= 350 \text{ ways}$$

iii) 3 belong to same class then either all 3 from 1^{st} class or all 3 from second class or all 3 from third class

$= {}^3C_3 + {}^4C_3 + {}^5C_3$

$= 1 + 4 + 10$

$= 15$ ways

Example 3.38 *How many automobile license plates can be made if each plate contains two different letters followed by three different digits. Solve the problem if first digit cannot be 0.*

Solution : I^{st} position is a letter and hence can be selected from 26 letters.

$= {}^{26}C_1$ ways

II^{nd} position is also a letter but different from I^{st} letter can be selected from 25 letters,

$= {}^{25}C_1$ ways

Now in digits,

I^{st} position can be selected from 10 digits

$= {}^{10}C_1$ ways

II^{nd} position can be selected from 9 digits as it is different from 1^{st} digit

$= {}^{9}C_1$ ways

And III^{rd} position can be selected from 8 digits as it is different from I^{st} and II^{nd} digit,

$= {}^{8}C_1$ ways

Hence, total number of license plates

$= 26 \times 25 \times 10 \times 9 \times 8$

$= 468000$

Now in license plate I^{st} digit cannot be zero, then I^{st} position can be selected out of 9 digits, i.e. in ${}^9C_1$ ways.

II^{nd} position can have zero, but one digit is already selected for 1^{st} position hence 2^{nd} digit can be selected in ${}^9C_1$ ways also III^{rd} digit can be selected in ${}^8C_1$ ways.

Hence, total number of auto license plates

$= 26 \times 25 \times 9 \times 9 \times 8 = 421200$

Example 3.39 *How many diagonals does an n sided regular polygon have ?*

Solution : A diagonal is formed by a line joining two vertices of a polygon. Hence, the number of lines joining two vertices of a polygon = Number of selecting 2 vertices out of n vertices

$= {}^nC_2$

$= \frac{n(n-1)}{2}$

Out of these $\frac{n(n-1)}{2}$ lines, n of them are forming the sides of the polygon. Therefore the number of diagonals.

$= \frac{n(n-1)}{2} - n$

$= \frac{n(n-3)}{2}$

Example 3.40 *How many rectangles are there in 8×8 chessboard ?*

Solution : To make a rectangle, we need two vertical lines and two horizontal lines. In an 8×8 chessboard, there are 9 vertical lines and 9 horizontal lines. Therefore, the number of rectangles

$= {}^9C_2 \times {}^9C_2$

$= \frac{9 \cdot 8}{2} \cdot \frac{9 \cdot 8}{2}$

$= 1296$

Example 3.41 *In how many ways can you take 5 cards, with atleast 2 aces, out of a well shuffled pack of 52 cards.*

Solution : First consider 5 cards, with exactly 2 aces.

For the two aces we have C(4, 2) possibilities and for the three cards we have C(48, 3) possibilities. The 5 cards can be chosen in $C(4, 2) \cdot C(48, 3)$ ways.

Similarly for 3 aces we have C(4, 3) possibilities and for the 2 cards we have C(48, 2) possibilities. Hence the 5 cards with 3 aces can be drawn in $C(4, 3) \cdot C(48, 2)$ ways and 4 aces can be chosen in $C(4, 4) \cdot C(48, 1)$ ways.

Hence total number of ways

$= C(4, 2) \cdot C(48, 3) + C(4, 3) \cdot C(48, 2) + C(4, 4) \cdot C(48, 1)$

Example 3.42 *There are 12 points in a plane of which 5 are collinear. Find the number of triangles that can be formed with vertices at these points.*

Solution : Since 5 points are collinear it means that the remaining 7 points are non collinear. These 7 points among themselves can form $C(7, 3) = \frac{7!}{4!\,3!} = 35$ triangles.

Taking any two collinear points and 1 non collinear point can form one triangle. Similarly taking any 2 non-collinear point and 1 linear point can form 1 triangle.

We can form $C(5, 2)\cdot C(7, 1) + C(5, 1)\cdot C(7, 2) = 70 + 105 = 175$ triangles with collinear and non-collinear points.

Hence total number of triangles

$$= 35 + 175 = 210$$

Example 3.43 *A school committee of 5 is to be formed from 12 students. How many committees can be formed if Saniya must be on the committee ?*

Solution : If Saniya must be on the committee, then 11 students are remaining out of which we have to choose 4.

Hence number of ways

$$= {}^{11}C_4 = \frac{11!}{7!\,4!}$$

$$= \frac{11\cdot 10\cdot 9\cdot 8}{4\cdot 3\cdot 2\cdot 1} = 330$$

Example 3.44 *There are 45 songs and you want to make a mix CD of 18 songs that must include 3 particular songs. How many different selections could you make ?*

Solution : Out of 45 songs, we have to choose 18 songs.

If 3 particular songs must be in the CD, then out of remaining 42 songs, we have to choose 15 songs, which can be selected in ${}^{42}C_{15}$ ways.

Example 3.45 *There are 8 parents and 43 students going on a school trip. Two groups are made, a large one with 30 students and 5 parents and a small group with 13 students and 3 parents.*

a) How many different ways can the parents be chosen for the small group ?

b) How many ways can the students be chosen for the large group if Shalaka, Saniya and Chintan must be in the small group ?

Solution : a) Out of 8 parents, 3 can be chosen in ${}^{8}C_3$ ways.

$$= \frac{8!}{5!\,3!} = \frac{8\cdot 7\cdot 6}{6} = 56$$

b) Since Shalaka, Saniya and Chintan are in small group, hence out of remaining 40 students 30 students can be chosen in ${}^{40}C_{30}$ ways.

Example 3.46 *If a student must select two courses from a group A, two courses from group B and one course from group C, how many combinations are there ?*

Group A	Group B	Group C
Maths 30 Chemistry 30 Physics 30 Biology 30	English 30 Social 30	Maths 31 Science 30 French 30

Solution : For group A, out of 4 courses, student must select 2 courses.

Hence number of ways to select 2 courses $= {}^{4}C_2$

Similarly in group B, out of 2 courses, student must select 2 course.

Hence number of ways $= {}^{2}C_2$ and for group C, 1 course out of 3 has to select, which can be done in ${}^{3}C_1$ ways.

Hence total number of ways

$$= {}^{4}C_2 \cdot {}^{2}C_2 \cdot {}^{3}C_1 = 6\cdot 1\cdot 3 = 18$$

Example 3.47 *A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if atleast 3 women are on the committee ?*

Solution : If atleast 3 women are on the committee, that means we can have a committee with 3 women or 4 women of 5 women.

i) If 3 women are in committee then number of ways to form committee

$$= {}^{7}C_3 \cdot {}^{4}C_2 = \frac{7\cdot 6\cdot 5}{6}\cdot 6 = 210$$

ii) If 4 women are in committee then number of ways to form committee

$$= {}^{7}C_4 \cdot {}^{4}C_1 = \frac{7\cdot 6\cdot 5}{6}\cdot 4 = 140$$

iii) If 5 women are in committee then number of ways $= {}^{7}C_5 = \frac{7\cdot 6}{2} = 21$

Hence if atleast 3 women are in committee then total number of ways

$$= 210 + 140 + 21$$

$$= 371$$

Example 3.48 *From a deck of 52 cards, a 5 card hand is dealt. How many distinct hands can be formed if there are atleast 2 queens ?*

Solution : The problem can be solved in the same way as previous problem. But there is another way to solve the problem.

Total possible ways for 5 card hand with no restriction = $^{52}C_5$

If no queen is included then out of remaining 48 non-queen cards we get 5.

Hence number of ways = $^{48}C_5$

If only 1 queen is included then out of 4 queen we get 1 and out of remaining 48 non-queen cards we get 4.

Hence number of ways = $^4C_1 \cdot {^{48}C_4}$

$\therefore$ Hand of 5 card can be formed in $^{52}C_5 - [^{48}C_5 + {^4C_1} \cdot {^{48}C_4}]$ ways.

Example 3.49 *A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists and 15 biologists. How many teams have*

a) Atleast 5 chemists ? b) Exactly 3 economists. c) 4 chemists, but no economists ?

d) Atleast 2 biologists ? e) 4 economists and 2 biologists ?

Solution : a) A team may have 5 chemists and 1 non-chemist or all 6 chemists. 5 chemists can be selected out of 10 chemists in $^{10}C_5$ ways and 1 non-chemist can be selected out of (5+8+15; i.e. 28) 28 non-chemists in $^{28}C_1$ ways.

Hence number of ways = $^{10}C_5 \cdot {^{28}C_1}$

Also committee may have all 6 chemists and number of ways for this selection

$$= {^{10}C_6}$$

Hence total number of ways in which atleast 5 chemists are selected

$$= {^{10}C_5} \cdot {^{28}C_1} + {^{10}C_6}$$

b) 3 economists can be selected out of 8 in $^8C_3$ ways and remaining 3 can be selected out of (10 + 5 + 15) 30 non-economists in $^{30}C_3$ ways.

Hence total number of ways = $^8C_3 \cdot {^{30}C_3}$

c) 4 chemists can be selected out of 10 chemists in $^{10}C_4$ ways and remaining 2 can be selected out of (5+ 15) 20 non-economists in $^{20}C_2$ ways.

Hence total number of ways = $^{10}C_4 \cdot {^{20}C_2}$

d) Total number of cases to select 6 people from 38 people = $^{38}C_6$.

If no biologist is selected for the committee this means all 6 are selected from remaining 23 people and hence number of ways = $^{23}C_6$.

If one biologist is selected for the committee then number of ways

$$= {^{23}C_5} \cdot {^{15}C_1}$$

Hence number of ways to select atleast 2 biologists

$$= {^{38}C_6} - ({^{23}C_6} + {^{23}C_5} \cdot {^{15}C_1})$$

e) 4 economists can be selected out of 8 in $^8C_4$ ways and 2 biologists can be selected out of 15 in $^{15}C_2$ ways.

Hence total number of ways = $^8C_4 \cdot {^{15}C_2}$

Example 3.50 *In how many ways can you choose one or more of 12 different candies?*

Solution : According to the question, we can take 1 candy, 2 candies and so on. The solution is found by adding all possible selections using,

$^{12}C_1 + {^{12}C_2} + \ldots + {^{12}C_{12}}$. However, this would take a long time.

Shortcut method

Each candy can be dealt within two ways. It can be chosen or not chosen. This will give 2 possibilities for the first candy, 2 for the second and so on. By multiplying the cases together we will get 2^{12}. Since the case of no candy being selected is not an option, we will have to subtract 1 from our answer.

There are $2^{12} - 1$ ways of selecting one or more candies.

3.5.1 Combination with Repetition

Now, we shall consider the case where we do not want order to matter, but will allow repetitions to occur.

Example : A = {a, b, c, d, e} and P = 6

Then (a, a, b, d, d, d), (b, b, b, c, d, e), (c, c, c, c, c, c) are combinations with repetition of 5 elements choose 6.

We can represent such combinations by means of a symbol with points and slashes.

{a, a, b, d, d, d} ⇔ ·· / · / / ··· /

{b, b, b, c, d, e} ⇔ / ··· / · / · / ·

{c, c, c, c, c, c} ⇔ / / ······ / /

Each symbol consists of 10 places with exactly 6 points and 4 slashes. With each such combinations there is just one symbol and with each symbol corresponds just one such combination.

We can build a symbol putting exactly 6 points in 10 places.

Afterwards, the spare places are filled with slashes.

In this manner, we can obtain any combination by moving the four slashes to occupy any of the 10 places.

Hence in the general case, the n-1 slashes should occupy any of the r+n-1 positions.

Hence the total number of ways to do this is C(n+ r− 1, n− 1).

Theorem :

The number of ways to fill r slots from n categories with repetition allowed is C(n+r−1, r) = C(n + r − 1, n − 1).

Example 3.51 *How many ways can one fill a box holding 100 pieces of candy from 30 different types of candy ?*

Solution : Here n = 100, r = 30

So there are C(100 + 30 − 1, 100)

$= C(129, 100)$

$= \frac{129!}{100!\,29!}$ different ways to fill the box.

Example 3.52 *In example 8.47, how many ways are there if one must have atleast 1 piece of each type ?*

Solution : Now, the slots are reduced to (100 − 30) as we have to select atleast 1 piece from each type, so there are

C(70+ 30− 1, 70) = C(99, 70)

$= \frac{99!}{70!\,29!}$ different ways to fill the box.

[Remark : When to use generalized combinations

Besides categorizing a problem based on its order and repetition requirements as a generalized combination, there are a couple of other characteristics which help us sort :

- In generalized combinations, having all the slots filled in by only selections from one category is allowed;
- It is possible to have more slots than categories.]

3.5.2 Integer Solutions to Equations

One other type of problem to be solved by the generalized combination formula is of the form :

How many non-negative integer solutions are there to the equation a + b + c + d = 100.

In this case, we could have 100 a's or 99 a's and 1 b, or 98 a's and 2 d's etc. We see that slots are 100 and we are ranging over 4 categories.

Therefore, there are

C(100 + 4 − 1, 4 − 1) = C(100 + 4 − 1, 100) as [C(n, r) = C(n, n − r)]

$= \frac{103!}{100!\,3!}$

Non-negative solutions to a + b + c + d = 100.

3.5.3 Integer Solutions with Restrictions

Example 3.53 *How many integer solutions are there to :*

$a + b + c + d = 5$.

when $a \geq 3$, $b \geq 0$, $c \geq 2$ and $d \geq 1$?

Solution : Now solution "strings" are

$\cdots a / b / \cdots c / \cdot d$,

where the a, b, c, d are the remaining numbers of each category to fill in the remaining slots.

However, the number of slots has effectively been reduced to 9 after accounting for a total of 6 restrictions.

Thus there are C(9 + 3, 9)

$= \frac{12!}{9!\,3!}$ solutions

3.5.4 More Integer Solutions and Restrictions

Example 3.54 *How many integer solutions are there to :*

$a + b + c + d = 15$.

When $a \geq -3$, $b \geq 0$, $c \geq -2$ and $d \geq -1$?

Solution : In this case, we alter the restrictions and equation so that the restrictions "go away". To do this, we need each restriction ≥ 0 and balance the number of slots accordingly.

Hence $a \geq -3+3$, $b \geq 0$, $c \geq -2+2$ and $d \geq -1+1$

yields a + b + c + d = 15 + 3 + 2 + 1 = 21

So, there are C(21 + 4 − 1, 21)

$$= \frac{24!}{21!\,3!} \text{ solutions}$$

3.5.5 Summary

Theorem 1 :

The number of integer solutions to $a_1 + a_2 + a_3 + \ldots + a_n = r$

when $a_1 \geq b_1,\ a_2 \geq b_2,\ a_3 \geq b_3, \ldots, a_n \geq b_n$ is

$C(n + r - 1 - b_1 - b_2 - b_3 - \ldots - b_n,\ r - b_1 - b_2 - b_3 - \ldots - b_n)$.

Theorem 2 :

The number of ways to select r things from n categories with b total restrictions on the r thing is

$$C(n + r - 1 - b,\ r - b)$$

Corollary 1 :

The number of ways to select r things from n categories with atleast 1 thing from each category is

$$C(r - 1,\ r - n) \quad (\text{set } b = n)$$

3.6 Generation of Permutations and Combinations

If we want to write down the n! permutations of n distinct objects, then for n = 1, 2 or 3 it is not difficult but when n is large, it is difficult to keep track of what we have written down and make sure we shall write down all the permutations with no omissions or repetition.

An interesting problem is to find a systematic procedure for generating all the n! permutations of a set of n distinct elements, with no omissions or repetitions.

Suppose from the initial permutation 1, 2, ... n, by using the Next-permutation procedure repeatedly, we may obtain all the permutations of 1, 2, ... n.

The last permutation is n, (n − 1), ... 3, 2, 1.

Procedure : Next-permutation

1. Given a permutation $a_1, a_2, \ldots a_n$ of 1, 2, ..., n.
2. Scan from right to left. Find the first m such that $a_m < a_{m+1}$.
3. $\alpha = \min \{a_k ; k = m+1, m+2, \ldots, n,\ a_k > a_m\}$;
4. The next-permutation is

$a_1, a_2, \ldots a_{m-1}, \alpha, x, x, x, \ldots,$

where x, x, x, ... are the remaining numbers arranged in the increasing order.

Example 1 : Let n = 6. Assume we are given the permutation 124653. Then m = 3, α = min{6, 5} = 5. The next permutation is 125346.

Example 2 : Let n = 6. Assume we are given the permutation 125346. Then m = 5, α = min {6} = 6. The next permutation is 125364.

Example 3 : Let n = 6. Assume we are given the permutation 125364. Then m = 4, α = min{6, 4} = 4. The next permutation is 125436.

Alternative method

In this method we generate all the n! permutations of {1, 2, 3, ..., n} by successively generating all the permutations of {1}, {1, 2}, {1, 2, 3} ...... upto {1, 2, ..., n − 1}. Let us see, how this is done.

Beginning with <1>, we obtain the permutations <1 2> and <2 1> placing 2 on either side of 1. Next, in the permutation <1 2>, we place 3 at the extreme ends and between 1 and 2 to obtain the permutation <1 2 3> <1 3 2> <3 1 2>.

Similarly from <2 1> we obtain <2 1 3>, <2 3 1>, <3 2 1>.

Proceeding in this manner, we obtain all permutations <1 2 3 .... n − 1> and then inserting n at both the ends and in the n − 2 gaps in between. We obtain all the n! permutations of {1, 2, ... n}.

For example, suppose we want to generate all the 24 permutations of {1, 2, 3, 4}. Then the sequential process, by which the permutations are generated is shown below.

<1>;

<1 2>, <2 1>;

<1 2 3>, <1, 3, 2>, <3 1 2>, <2 1 3>, <2 3 1>, <3 2 1>;

<1 2 3 4>, <1 2 4 3>, <1 4 2 3>, <4 1 2 3>;

<1 3 2 4>, < 1 3 4 2>, <1 4 3 2>, <4 1 3 2>;

<3 1 2 4>, <3 1 4 2>, <3 4 1 2>, <4 3 1 2>;

<2 1 3 4>, <2 1 4 3>, <2 4 1 3>, <4 2 1 3>;

<2 3 1 4>, <2 3 4 1>, <2 4 3 1>, <4 2 3 1>;

<3 2 1 4>, <3 2 4 1>, <3 4 2 1>, <4 3 2 1>;

Generate all k subsets of {1, 2, 3 .... n}, where k ≤ n.

duplicates.

We may always arrange the elements in a k-subset $\{a_1, a_2, \ldots, a_n\}$ in increas... order. Hence the first element $a_1 \le n-k+1$; the second element $a_2 \le n-k+2$, the ... element $a_m \le n-k+m$, the last element $a_k \le n-k+k$.

We will use β_m to denote the largest possible value of a_m (which is $n-k+m$).

Procedure : Next combination.

1. Given a subset $\{a_1, a_2, \ldots, a_k\}$.

   Assume $a_1 < a_2 < \ldots < a_k$.
2. The numbers not in the k-subset are denoted as $a_{k+1}, a_{k+2}, \ldots a_n$.
3. Scan from right to left. Find the first m such that $a_m < \beta_m$.
4. $\alpha = \{\min a_j ; j = m+1, m+2, \ldots k, k+1, \ldots n, a_m < a_j \le \beta_m\}$
5. The next k-subset is $\{a_1, a_2, \ldots, a_{m-1}, \alpha, b_{m+1}, b_{m+2} \ldots b_k\}$ where each ... is the smallest among the remaining numbers such that $b_{j-1} \le b_j \le \beta_j$ (where ... is defined as α).

To compute all the k-subsets, we start from the subset $\{1, 2, \ldots k\}$ and find successi... subsets. The last k-subset is $\{n-k+1, n-k+2, \ldots n\}$.

Example 1 :

Consider the subset {1, 3, 5, 6} of the set {1, 2, 3, 4, 5, 6}. Note that $\beta_1 = 3, \beta_2 = ...$ $\beta_3 = 5, \beta_4 = 6$. Scan from right to left. Choose m = 2 ($a_2 = 3 < \beta_2 = 4$). Then $\alpha = 4$. Now we have $\{1, 4, b_3, b_4\}$. For b_3, choose the smallest among {2, 3, 5, 6} that is greater than 4. Hence $b_3 = 5$. For b_4 choose the smallest among {2, 3, 6} that is greater than 5. Hence $b_4 = 6$.

Finally, we have {1, 4, 5, 6}.

Example 2

Consider the subset {1, 4, 5, 6} of the set { 1, 2, 3, 4, 5, 6}. Scan from right to left. Choose m = 1 ($a_1 = 1 < \beta_1 = 3$). Then $\alpha = 2$. Now we have $\{2, b_2, b_3, b_4\}$. For b_2, choose the smallest among {1, 3, 4, 5, 6} that is greater than 2. Hence $b_2 = 3$. For b_3, choose the smallest among {1, 4, 5, 6} that is greater than 3. Hence $b_3 = 4$. For b_4, choose the smallest among {1, 5, 6} that is greater than 4. Hence $b_4 = 5$. Finally we have {2, 3, 4, 5}.

3.7 Binomial Coefficients and Combinational Identities

Binomial theorem gives the expansion of $(a+b)^n$ where n is a positive integer and

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \ldots\ldots$$

$$+ {}^nC_r a^{n-r} b^r + \ldots + {}^nC_n a^0 b^n$$

The coefficients ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_r, {}^nC_n$ are known as binomial coefficients as they are present in expansion of $(a+b)^n$ using binomial theorem.

One method of proving the theorem is using mathematical induction.

We have another method, known as combinatorial method to prove the theorem.

For n = 3, consider

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

$$= aaa + aab + aba + abb + baa + bab + bba + bbb$$

The first term aaa occurs, when we select 'a' from each factor (a + b) and there is only one such term, i.e. a^3 in the final expansion. There are three terms aab, aba, baa in which 'a' appears twice and 'b' only once. The same is the case with b twice and a only once.

The last term bbb occurs only once, like aaa.

Putting the above observations differently, we note

i) There are two choices a or b for each term in the product and so there are $2 \times 2 \times 2 = 8$ formal products.

ii) In the 8 formal products, there are 3 in which a appears twice and same is in the case for b.

iii) There is only a single product in which a (as well as b) has all the three choices.

Using the above combinatorial arguments we conclude that

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

For the general case $(a+b)^n$, we can argue similarly. Each product in the expansion contains n factors. Suppose we want to find how many times the product $a^r b^{n-r}$ appears in the expansion. Since a must have r choices, these can be found in ${}^nC_r$ ways. The number of choices is exactly the same for b to have n − r choices. Hence it follows that

$${}^nC_r = {}^nC_{n-r}$$

The binomial coefficient ${}^nC_r$ can also be interpreted differently using set theory and this interpretation is found very often useful.

If we have n persons, out of which we want to form a committee of r persons. This is equivalent to finding r-element subsets of a set having n elements and this number is ${}^nC_r$.

Some results regarding combinations (i.e. binomial coefficients) which are considered as combinational identitites.

1. ${}^{n}C_{r} = {}^{n}C_{n-r}$

Proof : R.H.S. ${}^{n}C_{n-r} = \dfrac{n!}{(n-r)!\,(n-(n-r))!}$

$$= \frac{n!}{(n-r)!\,r!}$$

$$= {}^{n}C_{r}$$

Hence $\quad {}^{10}C_{6} = {}^{10}C_{10-6}$

i.e. $\quad {}^{10}C_{6} = {}^{10}C_{4}$

$${}^{12}C_{3} = {}^{12}C_{12-3}$$

$$= {}^{12}C_{9}$$

Remark

i) ${}^{n}C_{r}$ has a meaning only when $0 \le r \le n$.

${}^{n}C_{n-r}$ has a meaning only when $0 \le n-r \le n$.

ii) ${}^{n}C_{r}$ and ${}^{n}C_{n-r}$ are called complementary combinations.

2. ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r},\ 1 \le r \le n$

Proof : Using algebraic method,

$$\frac{n!}{r!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)!}$$

$$= \frac{n!}{r(r-1)!(n-r)!} + \frac{n!}{(r-1)!\,[(n-r+1)\,((n-r)!)]}$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{1}{r} + \frac{1}{n-r+1}\right]$$

$$= \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{n+1}{r(n-r+1)}\right]$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = {}^{n+1}C_{r}$$

Proof using combinatorics :

Let A be a set containing n elements choose $b \notin A$ and form the set $B = A \cup \{b\}$.

Then $C(n+1, r)$ is the number of r element subsets of B. These can be divided into two disjoint classes.

i) Subsets of B not containing b.

ii) Subsets of B containing b.

The subsets of class 1 are just r element subsets of A and those of class 2 consist of a $r-1$ element subset of A together with b. Number of subsets of the first type is $C(n, r)$ and that of the second type is $(n, r-1)$.

Hence $C(n+1, r) = C(n, r-1) + C(n, r)$

3. ${}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots + {}^{n}C_{n} = 2^{n}$

The sum of binomial coefficients is equal to 2^{n}.

Proof : Using binomial theorem,

$$(a+b)^{n} = {}^{n}C_{0}a^{n} + {}^{n}C_{1}a^{n-1}b + {}^{n}C_{2}a^{n-2}b^{2} + \ldots\ldots + {}^{n}C_{n}b^{n}$$

Substitute $\quad a = b = 1$

$$(1+1)^{n} = {}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots + {}^{n}C_{n}$$

$$\therefore \quad {}^{n}C_{0} + {}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots + {}^{n}C_{n} = 2^{n}$$

Corollary : Since ${}^{n}C_{0} = 1$

$$1 + {}^{n}C_{1} + {}^{n}C_{2} + \ldots + {}^{n}C_{n} = 2^{n}$$

$$\therefore \quad {}^{n}C_{1} + {}^{n}C_{2} + \ldots\ldots + {}^{n}C_{n} = 2^{n} - 1$$

Proof by combinatorics method :

Let A be a set containing n elements. Then the number of subsets of $A = 2^{n}$.

Number of subsets of A containing no element is 1 (i.e. empty set) or ${}^{n}C_{0}$.

Number of subsets of A containing 1 element are n or ${}^{n}C_{1}$.

Similarly number of subsets of A containing 2, 3, 4, ... elements will be ${}^{n}C_{2}, {}^{n}C_{3}, {}^{n}C_{4}$ .... and so on.

Hence total number of subsets of A,

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$$

i.e.

4. $${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots = 2^{(n-1)}$$

Proof : Using binomial theorem,

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots\ldots + {}^nC_n a^0 b^n$$

Let a = 1 and b = −1

Then $(1-1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + {}^nC_4 - {}^nC_5 + {}^nC_6 \ldots\ldots$

$\therefore \quad 0 = [{}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots] - [{}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots]$

$\Rightarrow \quad {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots\ldots$

Also $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots + {}^nC_n = 2^n$

$\Rightarrow [{}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots] + [{}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots] = 2^n$

$\Rightarrow \quad 2[{}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots] = 2^n$

$\Rightarrow \quad {}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots\ldots = 2^{(n-1)}$

Hence $\quad [{}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots] = [{}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots] = 2^{(n-1)}$

5. $C(n, r) \cdot C(r, m) = C(n, m) \cdot C(n-m, r-m), \quad 1 \le m \le r \le n$

Proof :

$$\frac{n!}{r!\,(n-r)!} \cdot \frac{r!}{m!\,(r-m)!} = \frac{n!}{m!\,(n-m)!} \cdot \frac{(n-m)!}{(n-r)!\,(r-m)!}$$

$$= C(n, m) \cdot C(n-m, r-m)$$

Proof using combinatorics

LHS gives the number of ways to select r element subsets of a set of n elements and then to select m element subsets from each of these r subsets.

This can be equivalently substituted by selecting m-element subsets from the set of n elements and then remaining r − m element subsets from remaining n − m elements.

Example 3.55 *Prove that* $({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$.

Solution : Using equality,

$$(1+x)^n \cdot (1+x)^n = (1+x)^{2n}$$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \ldots\ldots + {}^nC_n x^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^r$$

$$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \ldots\ldots + {}^{2n}C_{2n} x^{2n}$$

$$= \sum_{k=0}^{2n} {}^{2n}C_k\, x^k$$

Coefficient of x^n should be equal on both the sides.

Hence

$${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \ldots + {}^nC_n \cdot {}^nC_0 = {}^{2n}C_n$$

But ${}^nC_0 = {}^nC_n,\ {}^nC_1 = {}^nC_{n-1},\ {}^nC_2 = {}^nC_{n-2}$

....

$\Rightarrow ({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n$

Example 3.56 *Show that* ${}^nC_1 + 6({}^nC_2) + 6({}^nC_3) = n^3$.

Solution : ${}^nC_1 + 6({}^nC_2) + 6({}^nC_3)$

$$= \frac{n!}{1!\,(n-1)!} + \frac{6(n!)}{2!\,(n-2)!} + \frac{6(n!)}{3!\,(n-3)!}$$

$$= \frac{n!}{(n-3)!}\left[\frac{1}{(n-1)(n-2)} + \frac{6}{2 \cdot (n-2)} + 1\right]$$

$$= \frac{n!}{(n-3)!}\left[\frac{1 + 3(n-1) + (n-1)(n-2)}{(n-1)(n-2)}\right]$$

$$= n(n-1)(n-2)\left[\frac{1 + 3n - 3 + n^2 - 3n + 2}{(n-1)(n-2)}\right]$$

$$= n[n^2]$$

$$= n^3$$

Example 3.57 *If* $C(n, 3) + C(n+2, 3) = P(n, 3)$, *find* n.

Solution : $\quad {}^nC_3 + {}^{n+2}C_3 = {}^nP_3$

$$\frac{n!}{3!\,(n-3)!}+\frac{(n+2)!}{3!\,(n-1)!} = \frac{n!}{(n-3)!}$$

$$\Rightarrow \quad \frac{n\,(n-1)\,(n-2)}{6}+\frac{(n+2)\,(n+1)\,(n)}{6} = n(n-1)\,(n-2)$$

$$\Rightarrow \quad (n-1)\,(n-2)+(n+2)\,(n+1) = 6(n-1)\,(n-2)$$

$$\Rightarrow \quad (n+2)\,(n+1) = 5(n-1)\,(n-2)$$

$$\Rightarrow \quad n^2+3n+2 = 5(n^2-3n+2)$$

$$\Rightarrow \quad 4n^2-18n+8 = 0$$

$$\Rightarrow \quad 2n^2-9n+4 = 0$$

$$\Rightarrow \quad (2n-1)\,(n-4) = 0$$

$$\Rightarrow \quad n = \frac{1}{2}, 4$$

$$\Rightarrow \quad n \neq \frac{1}{2}$$

But

Hence $$\boxed{n = 4}$$

□□□

4 Propositional Logic

Syllabus

Definition, Statements & Notation, Truth Values, Connectives, Statement Formulas & Truth Tables, Well-formed Formulas, Tautologies, Equivalence of Formulas, Duality Law, Tautological Implications, Examples

Contents

4.1 Logic

Logic is analysis of language, which consists of signs. Logic is a set of rules (or axioms) which we can use to draw valid conclusions.

4.2 Statements (or Propositions)

Statements are kinds of sentences which we have to use to convey our thoughts to others. A sentence is a **statement** or **proposition** if it is possible to say whether what is conveyed by the sentence is true or false.

i.e. A sentence of which it is meaningful, to say, whether it is true or false but not **both** is called a statement or proposition.

We denote statements by letters p, q, r, s.....

Examples

1. Pune is capital of India.
2. Mars is a planet.
3. $9 > 13$.
4. $y + 8 = 12$
5. $6 \in \{1, 2, 3, 4\}$
6. $x \in A$.
7. There are 12 months in a year.

Examples 2 and 7 are true statements.

Examples 1, 3, 5 are false statements. In example 4, if we write 4 in place of y, it becomes a true sentence and if y is not 4 then it is a false sentence. Similarly in 6 we cannot say whether this sentence is true or not unless we are told what are the elements of A.

Thus only those sentences are statements about which we can say whether these are true or false.

Definition

A **statement** is a **declarative sentence** with a definite truth value.

Hence 1, 2, 3, 5, 7 are statements.

4 is not a statement since its truth value depends on the value of y. If y is 4 the sentence is true, if $y \neq 4$ then the sentence is false.

Also 6 is not a statement since its truth value depends on set A, if $A = \{x, y, z, t\}$ then 6 is true but if $A = \{1, 2, 3, 4\}$ then 6 is false.

4.3 Open Statement

In example ④ above if we put $y = 4$, it becomes a true statement, if we take value of $y \neq 4$, it becomes false. Such statements are open statements. Thus if a mathematical sentence is neither true nor false it is called open sentence.

Definition

An open statement is a sentence that contains one or more variables such that when certain values are substituted for the variables, we get statements.

Examples

1. $x + 7 = 9$
2. $3x + 2 > 8$
3. $x \in A$

4.4 Truth Value of a Statement

Statement has a definite truth value which is either true or false. True values are denoted by (T) and false values are denoted by (F).

4.5 Truth Table

A table giving all possible truth values of a statement is called truth table.

Exercise 4.1

1. *Which of the following statements are true and which are false ?*
   a) $9 < 12$
   b) *2 is a prime number.*
   c) *41 is a composite number.*
   d) $2 + 5 = 3 + 9$
2. *What type of the sentence is*
   $x + 8 = 17$?
3. *For what value of x following sentences will become true statements ?*
   a) $3x + 9 = 15$ b) $x + 6 = 8$
   c) $x + 1 > 5$ d) $x + 2 < 8$
   e) $5x \geq 25$ f) $5x \leq 25$
4. *Which of the following open statements are true ?*
   a) $x + 4 = 6$ *when* $x = 2$ b) $x + 4 \neq 6$ *when* $x = 2$
   c) $x + 5 \neq 8$ *when* $x = 3$ d) $2x + 4y = 14$ *when* $x = 1, y = 3$
   e) $3x + 5y = 11$ *when* $x = 0, y = 2$ f) $5 \in \{4, 2, x\}$ *when* $x = 5$

5. *Define the following and illustrate each by an example.*

   a) Statement

   b) Open statement

Answers

1. a) True b) True c) False d) False
2. Open statement.
3. a) $x = 2$ b) $x = 2$ c) $x \geq 5$

   d) $x < 6$ or $x \leq 5$ e) $x \geq 5$ f) $x \leq 5$
4. a) True b) False c) False

   d) True e) False f) True.

4.6 Logical Connectives

Every statement must be either true or false but not both.

If $\exists$ two or more statements, they can be combined to produce a new statement. These statements are called compound statements.

To combine statements we use following symbols.

4.6.1 Conjunction ('$\wedge$' or 'and')

If p and q are statements, the compound statement '$p \wedge q$' is 'p and q' and is called 'p conjunction q' or 'p meet q'.

'$\wedge$' denotes 'and' and is known as conjunction.

Examples

1. Let us consider

   p : Shalaka is in senior K.G.

   q : Shalaka likes drawing,

   then $p \wedge q$ is the statement

   Shalaka is in senior K.G. and likes drawing.

Definition

When two or more statements are combined by the word 'and' the compound statement is known as 'conjunction'.

Examples

Let us consider the statements.

2. p : 3 > 2

   q : 9 is an odd number,

   then p ∧ q is the statement

   3 > 2 and 9 is an odd number.

3. p : 25 is divisible by 5.

   q : Shalaka likes swimming,

   then p ∧ q is the statement

   25 is divisible by 5 and Shalaka likes swimming.

[Remark

i. If there is no connection between two statements still we can combine them in logic.

ii. 'But' and 'while' are treated as equivalent words to 'and'.

iii. Conjunction is a binary connective as in conjunction we need atleast two statements to combine.]

1. If p is true and q is true then p ∧ q means the resulting statement is true.
2. If p is false and q is false then p ∧ q is false.
3. If p is false and q is true then p ∧ q is false.
4. If p is true and q is false then p ∧ q is false.

Truth table for p ∧ q

p	q	p ∧ q
T	T	T
F	F	F
T	F	F
F	T	F

p ∧ q is true only when both are true otherwise false.

4.6.2 Disjunction ('∨' or 'or')

When two or more statements are combined by the word 'or', the compound statement is known as 'disjunction'.

The symbol 'p ∨ q' is read as 'p or q' or 'p disjunction q' or 'p join q'.

Examples

Let us consider the statements.

1. p : I will purchase a dress.
   q : I will purchase a book,
   then $p \vee q$ is the statement
   I will purchase a dress or I will purchase a book.

2. p : a is equal to 5.
   q : b is equal to 7,
   then $p \vee q$ is the statement
   a is equal to 5 or b is equal to 7.

1. If p is true and q is true,
   then $p \vee q$ is true.
2. If p is true and q is false,
   then $p \vee q$ is true.
3. If p is false and q is true,
   then $p \vee q$ is true.
4. If p is false and q is false,
   then $p \vee q$ is false.

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Remark

i. '$p \vee q$' is true if either p is true or q is true or both p and q are true and $p \vee q$ is false if both p and q are false.

ii. Disjunction is a binary connective.

iii. The exclusive disjunction or exclusive 'OR' of two propositions p and q is the statement. 'Either p is true or q is true, but both are not true.'

Either p is true or q is true, but both are not true, we denote this by $p \oplus q$.

iv. Inclusive disjunction or inclusive 'OR' of two propositions p and q is the statement. 'Either p is true or q is true or both true.' i.e. $p \vee q$.

v. Truth table for $p \oplus q$ [exclusive 'OR']

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Example for exclusive 'OR'

I shall go for walk or I shall watch TV. Here the connective 'or' is used in exclusive sense.

i.e. Either one or the other is possible at a time but both are not possible.

4.6.3 Negation (~)

Let p be any statement then negation of p is denoted by '~p' (or $\tilde{p}$) is read as 'not p'.

If p is true then ~ p is false.

If p is false then ~ p is true.

Truth table

p	~ p
T	F
F	T

~ p is a unary connective as only one statement is required to form negation.

Examples

1. If p is the statement 'Ram is intelligent boy',
   then ~ p is the statement.
   'Ram is not intelligent boy.'
   or 'It is not the case that Ram is intelligent boy.'
2. If p is the statement 'I like to read',
   then ~ p is the statement.
   'I don't like to read'
   or 'It is not the case that I like to read.'

4.6.4 Conditional Connectives (If .... then ....) or Implication

Consider the proposition 'If Akash gets 90 % or more in 12^{th} examination, then he will get admission for engineering course.'

We can write this statement as

'If p then q'

p : Akash gets 90 % or more in 12^{th} examination.

q : Akash will get admission for engineering course.

This is an example of implication of q by p.

Definition

Let p and q be two propositions, then the statement

'if p then q' is denoted by $p \rightarrow q$ (p implies q) or 'p is sufficient for q' or 'p only if q' or 'q if p' or 'q is necessary for p'.

p is known as hypothesis or antecedent and q is known as conclusion or consequent.

A statement or proposition of the form $p \rightarrow q$ is called a conditional statement or conditional proposition.

Example

'If a is in $\Re$, then a is in $\mathbb{C}$ (complex number) the hypothesis is $a \in \Re$ and the conclusion is $a \in \mathbb{C}$.

This can be expressed as $a \in \Re \rightarrow a \in \mathbb{C}$

1) If p is true, q is true then

$p \rightarrow q$ is true.

2) If p is true, q is false then

$p \rightarrow q$ is false.

3) If p is false, q is true then

$p \rightarrow q$ is true.

4) If p is false, q is false then

$p \rightarrow q$ is true.

Truth table for $p \rightarrow q$

p	q	$p \rightarrow q$
T	T	T

T	F	F
F	T	T
F	F	T

Examples for conditional connectives

1. p : The function is differentiable.

   q : The function is continuous.

   If p then q

   i.e. $p \to q$

   or p is sufficient for q.

2. p : Weather is good.

   q : Match will take place.

   If weather is good then the match will take place.

   $p \to q$

3. p : (F, +, ·) is a field.

   q : (I, +, ·) is an integral domain.

   r : (R, +, ·) is a ring.

   (i) $p \to q$

   Every field is an integral domain.

   (ii) $p \to r$

   Every field is a ring.

   (iii) $q \to r$

   Every integral domain is a ring.

Remark

If p, then q

q if p

p only if q,

p is sufficient for q

q is necessary for p

are all equivalent to the statement.

$p \to q$.

Remark

1. Converse of $p \rightarrow q$ is $q \rightarrow p$.
2. Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

4.6.5 Biconditional (or Double Implication or Equivalence)

If p and q are two propositions, then '$p \rightarrow q$' and '$q \rightarrow p$' is called biconditional or double implication. It is written as

'$p \leftrightarrow q$'

and it is known as

'p if and only if q'

'if and only if' is shortened as 'iff'. $\leftrightarrow$ is also known as 'implies and is implied'.

We also say 'p is necessary and sufficient for q'.

$p \leftrightarrow q$ is true when truth values of p and q are same, i.e. either both true or both false and false otherwise.

Truth table for $p \leftrightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$ $(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example

In a triangle ABC,

$p : \angle C = 90^\circ$

$q : (AC)^2 + (BC)^2 = (AB)^2$

If p then q

i.e. $p \rightarrow q$ ... (1) [By Pythagoras theorem]

Also if q then p

i.e. $q \rightarrow p$ ... (2) [If $(AC)^2 + (BC)^2 = (AB)^2$ then $\angle C = 90^\circ$]

From equations (1) and (2),

$p \leftrightarrow q$

4.6.6 Contrapositive of an Implication

The implication $\sim q \rightarrow \sim p$ is called the contrapositive of the implication $p \rightarrow q$.
i.e. implication $p \rightarrow q$ and $\sim q \rightarrow \sim p$ are the contrapositive of each other.

Example

i) If in a triangle ABC, $\angle A = 90°$ then $(AB)^2 + (AC)^2 = (BC)^2$.

ii) If in a triangle ABC, $(AB)^2 + (AC)^2 \neq (BC)^2$ then $\angle A \neq 90°$.

Above two implications are the contrapositive of each other.

1) If p is true, q is true then $p \rightarrow q$ is true and $\sim q$ is false, $\sim p$ is false but $\sim q \rightarrow \sim p$ is true.
2) If p is true and q is false then $p \rightarrow q$ is false, $\sim q$ is true and $\sim p$ is false but $\sim q \rightarrow \sim p$ is false.
3) If p is false, q is true, then $p \rightarrow q$ is true, $\sim q$ is false, $\sim p$ is true but $\sim q \rightarrow \sim p$ is true.
4) If p is false, q is false, then $p \rightarrow q$ is true, $\sim q$ is true, $\sim p$ is true and $\sim q \rightarrow \sim p$ is true.

The following table gives the truth value of the implications $p \rightarrow q$ and $\sim q \rightarrow \sim p$.

Truth table for $p \rightarrow q$ and $\sim q \rightarrow \sim p$

p	q	$p \rightarrow q$	$\sim q$	$\sim p$	$\sim q \rightarrow \sim p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example 4.1 *Express following statements in propositional form :*

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi) If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey.

Solution : i) p : There are many clouds in the sky.

q : It rain.

$\therefore$ $p \wedge \sim q$

ii) p : I will get first class.

q : I study well.

r : Score above 80 in mathematics.

$\therefore$ $p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap.

q : Softwares are costly.

$\therefore$ $p \wedge q$.

iv) p : It is very hot.

q : It is very humid.

r : Ramesh is having heart problem.

$\therefore$ $(p \wedge q) \vee r$

v) p : In small restaurant food is good.

q : Service is poor.

$\therefore$ $p \wedge q$

vi) p : I finish my submission before 5.00 pm.

q : It is very hot.

r : I will go.

s : I will play a game of hockey.

$\therefore$ $(p \wedge \sim q) \rightarrow (r \wedge s)$

Example 4.2 *Let p denote the statement, 'The material is interesting'.*

q denote the statement, 'The exercises are challenging', and

r denote the statement, 'The course is enjoyable'.

Write the following statements in symbolic form :

i) The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii) Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging, then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable.

Solution : i) $p \wedge q$

ii) $(p \rightarrow q) \wedge (q \rightarrow p)$

iii) $p \oplus \sim q$

iv) $(\sim p \wedge \sim q) \rightarrow \sim r$

v) $\sim p \wedge \sim q \wedge \sim r$.

Example 4.3 *Express the contrapositive, converse, inverse and negation forms of the conditional statement given below :*

'if x is rational, then x is real.'

Solution : Let p : x is rational.

q : x is real.

Symbolic form : $p \rightarrow q$

Contrapositive : $(\sim q \rightarrow \sim p)$

If x is not real, then x is not rational.

Converse : $(q \rightarrow p)$

If x is real then x is rational.

Inverse : $(\sim p \rightarrow \sim q)$

If x is not rational, then x is not real.

Negation : $\sim (p \rightarrow q)$

$\equiv \sim (p \vee \sim q)$

$\equiv \sim p \wedge \sim \sim q$

$\equiv \sim p \wedge q$

Example 4.4 *Express the contrapositive, converse and inverse forms of the following statement.'if $3 < b$ and $1 + 1 = 2$, then $\sin \frac{\pi}{3} = \frac{1}{2}$'.*

Solution : $p : 3 < b$

$q : 1 + 1 = 2$

$r : \sin \frac{\pi}{3} = \frac{1}{2}$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim (p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin \frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1 + 1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin \frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim (p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$

i.e. if $3 \geq b$ or $1 + 1 \neq 2$ then $\sin \frac{\pi}{3} \neq \frac{1}{2}$

Exercise 4.2

1. *If 'p' stands for the statement 'I read book' and 'q' stands for 'I go for swimming', translate the following in words :*

   a) $p \wedge q$ *b)* $p \vee q$

   c) $\sim p \vee q$ *d)* $p \wedge (\sim q)$

2. *If 'p' stands for the statement 'Ramesh is a player' and 'q' stands for 'Mohan is a strong boy', translate the following in words :*

   a) $p \wedge q$ *b)* $\sim p \wedge q$

   c) $p \wedge (\sim q)$ *d)* $\sim p \wedge \sim q$

   e) $\sim (p \wedge q)$ *f)* $\sim p \leftrightarrow q$

   g) $p \rightarrow \sim q$

3. *If 'p' stands for 'I run fast' and 'q' stands for 'I shall win', express the following symbolically :*

   a) I do not run fast.

   b) If I run fast, I shall win.

   c) I run fast or I shall not win.

   d) I run fast and I shall win.

   e) I neither run fast nor I shall win.

   f) I shall win if and only if I run fast.

4. *If p stands for 'it is cold' and q stands for 'it is raining' translate the following into symbols :*

   a) It is cold or it is raining.

   b) It is cold and it is raining.

   c) It is cold but it is not raining.

   d) It is not cold but it is raining.

   e) If it is not cold then it is not raining.

   f) Sufficient condition for rain is that it is cold.

   g) A necessary condition for raining is that it be cold.

   h) A necessary and sufficient condition for raining is that it be cold.

Answers

1. a) I read book and I go for swimming.
   b) I read book or I go for swimming.
   c) Either I do not read book or I go for swimming.
   d) I read book and I do not go for swimming.
2. a) Ramesh is a player and Mohan is a strong boy.
   b) Ramesh is not a player and Mohan is a strong boy.
   c) Ramesh is a player and Mohan is not a strong boy.
   d) Neither Ramesh is a player nor Mohan is a strong boy.
   e) This is not the case that Ramesh is a player and Mohan is a strong boy.
   f) Ramesh is not a player if and only if Mohan is a strong boy.
   g) If Ramesh is a player then Mohan is not a strong boy.
3. a) $\sim p$ b) $p \rightarrow q$
   c) $p \vee (\sim q)$ d) $p \wedge q$
   e) $\sim p \wedge \sim q$ f) $q \leftrightarrow p$
4. a) $p \vee q$ b) $p \wedge q$
   c) $p \wedge (\sim q)$ d) $(\sim p) \wedge q$
   e) $\sim p \rightarrow \sim q$ f) $q \rightarrow p$
   g) $p \rightarrow q$ h) $p \leftrightarrow q$

4.7 Tautology

A tautology is a proposition which is true for all truth values of its sub-propositions.

In other words a proposition is a tautology if it is always **true** for all assignments of truth values.

4.8 Contradiction

A proposition is a contradiction if it is always **false** for all assignments of truth values.

Remark : A proposition which is neither a tautology nor a contradiction is called a **contingency.**

Example 4.5 *Show that $p \rightarrow p$ is a tautology.*

Solution :

p	p	p → p
T	T	T
F	F	T

Since in $p \rightarrow p$ all the truth values are true (T), hence $p \rightarrow p$ is a tautology.

Example 4.6 *Show that $p \vee (\sim p)$ is a tautology.*

Solution :

p	~ p	p ∨ (~ p)
T	F	T
F	T	T

Since in $p \vee (\sim p)$, all the truth values are true (T), hence $p \vee (\sim p)$ is a tautology.

Example 4.7 *Show that $\sim (p \wedge (\sim p))$ is a tautology.*

Solution :

p	~ p	p ∧ (~ p)	~ (p ∧ (~ p))
T	F	F	T
F	T	F	T

Hence $\sim (p \wedge (\sim p))$ is a tautology.

Remark : $p \wedge (\sim p)$ is a **contradiction** as all the truth values are false (F).

Example 4.8 *Show that $(p \rightarrow q) \leftrightarrow (\sim p) \vee q$ is a tautology.*

Solution :

p	q	~ p	p → q	(~ p) ∨ q	(p → q) ↔ (~ p) ∨ q
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Hence $(p \rightarrow q) \leftrightarrow (\sim p) \vee q$ is a tautology.

Example 4.9 *Construct truth table for (question 1 to question 5) to find if each of the following is a tautology, contradiction or contingency.*

1) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.

Solution : Consider the truth table.

p	q	r	$(p \rightarrow q)$	$(p \rightarrow r)$	$(q \rightarrow r)$	$(p \rightarrow (q \rightarrow r))$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$
T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	F	T	T	T	T	T	T

In the last column all truth values are (T), hence

$(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$ is a tautology.

2) $((p \vee q) \wedge \sim p) \rightarrow q$

Solution : Consider the truth table.

p	q	$\sim p$	$p \vee q$	$(p \vee q) \wedge \sim p$	$((p \vee q) \wedge \sim p) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	F	F	T

Hence $((p \vee q) \wedge \sim p) \rightarrow q$ is a tautology.

3) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Solution : Consider the truth table.

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$(p \rightarrow r)$	$[(p \rightarrow q) \wedge (q \wedge r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	F	T	T
T	T	F	T	F	F	F	T

T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	F	T	T	T	T	T

Hence $[(p \to q) \wedge (q \to r)] \to (p \to r)$ is a tautology.

4) $(p \wedge q) \wedge \sim (p \vee q)$

Solution : Consider the truth table.

p	**q**	**$p \wedge q$**	**$p \vee q$**	**$\sim (p \vee q)$**	**$(p \wedge q) \wedge \sim (p \vee q)$**
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

5) $(p \to q) \leftrightarrow (q \vee \sim p)$

Solution : Consider the truth table.

p	**q**	**$\sim p$**	**$p \to q$**	**$q \vee \sim p$**	**$(p \to q) \to (q \vee \sim p)$**	**$(q \vee \sim p) \to (p \to q)$**	**$(p \to q) \leftrightarrow (q \vee \sim p)$**
T	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	T	T	T	T	T
F	F	T	T	T	T	T	T

Hence $(p \to q) \leftrightarrow (q \vee \sim p)$ is a tautology.

Example 4.10 *Verify that the proposition $p \vee \sim (p \wedge q)$ is a tautology.*

Solution :

p	**q**	**$p \wedge q$**	**$\sim (p \wedge q)$**	**$p \vee \sim (p \wedge q)$**
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Hence $p \vee \sim (p \wedge q)$ is a tautology.

4.9 Precedence Rule

While dealing with operations on numbers, we realize the importance of BODMAS rule. According to this rule when calculating the value of an arithmetic expression, we first calculate the value of the Bracketed part then apply **of, division, multiplication, addition** and **subtraction** in this order.

Similarly while calculating the truth value of compound propositions involving more than one connective, we have a similar **convention** which tell us which connective to apply first.

Suppose we did not have an order of preference and want to find the truth of $\sim p \vee q$. Some may consider the value of $(\sim p) \vee q$ and some may consider $\sim (p \vee q)$. The truth values can be different in these cases.

For example :

If p and q are both true, then $(\sim p) \vee q$ is true but $\sim (p \vee q)$ is false. So, for the purpose of unambiguity, we agree to such an order or rule.

The rule of precedence : The order of preference in which the connectives are applied in a formula of propositions that has no brackets is

i) $\sim$

ii) $\wedge$

iii) $\vee$ and $\oplus$

iv) $\rightarrow$ and $\leftrightarrow$

Remark :

1. '$\vee$' includes 'exclusive or' and 'inclusive or' both.
2. '$\vee$' and '$\oplus$' which are both third in order of preference. If both these appear in a statement, we first apply the leftmost one. The same applies to the $\rightarrow$ and $\leftrightarrow$.

4.10 Logical Equivalence

In mathematics, as in ordinary language, there can be several ways of saying the same thing.

Definition

Two propositions A and B are logically equivalent if and only if they have the same truth value for every choice of truth values of simple propositions involved in them.

We denote this fact by $A \equiv B$.

In other words two propositions A and B are logically equivalent if and only if $A \leftrightarrow B$ is a tautology.

Logically equivalent using truth tables

Example 4.11 *Prove that* $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$.

Solution : Consider the truth table.

p	q	~ p	p ∨ q	(p ∨ q) ∧ ~ p	~
T	T	F	T	F	
T	F	F	T	F	
F	T	T	T	T	
F	F	T	F	F	

From the table, truth values of $(p \vee q) \wedge \sim p$ and $\sim p \wedge q$ are same for each p and q. Hence $(p \vee q) \wedge \sim p$ is equivalent to $\sim p \wedge q$.

i.e. $(p \vee q) \wedge \sim p \equiv \sim p \wedge q$

Example 4.12 *Prove that* $(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$.

Solution : Consider truth table.

p	q	r	p → q	q → r	p → r	p → (q → r)	(p → q) → (
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

In last two columns, truth values are same for each choice of truth values of p, q and r hence

$$p \rightarrow (q \rightarrow r) \equiv (p \rightarrow q) \rightarrow (p \rightarrow r)$$

or $(p \rightarrow (q \rightarrow r)) \Rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

4.10.1 Duality Law

Two formulas A and A* are said to be the duals of each other if either one can obtained from the other by replacing '∧' by '∨' and '∨' by '∧'. The connectives '∧' and are also called dual of each other.

Logical identities

1. DeMorgan's laws

i) $\sim(p \vee q) \equiv (\sim p \wedge \sim q)$

ii) $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$ [Dual of (i)]

2. Associative laws

i) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

ii) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$ [Dual of (i)]

3. Commutative laws

i) $p \vee q \equiv q \vee p$

ii) $p \wedge q \equiv q \wedge p$ [Dual of (i)]

4. Idempotent laws

i) $p \vee p \equiv p$

ii) $p \wedge p \equiv p$ [Dual of (i)]

5. Double negation

$\sim(\sim p) \equiv p$

6. Distributive laws

i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ [Dual of (i)]

7. Absorption laws

i) $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$ [Dual of (i)]

Logically equivalent

Example 4.13 *DeMorgan's laws*

i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Solution : i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table.

p	q	~ p	~ q	p ∨ q	~ (p ∨ q)	~ p ∧ ~ q
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q. Hence

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Consider the table.

p	q	~p	~q	p ∧ q	~(p ∧ q)	~p ∨ ~q
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

Example 4.14 *Absorption laws*

i) $p \vee (p \wedge q) \equiv p$

ii) $p \wedge (p \vee q) \equiv p$

Solution : i) $p \vee (p \wedge q) \equiv p$

Consider the table.

p	q	p ∧ q	p ∨ (p ∧ q)	p
T	T	T	T	T
T	F	F	T	T
F	T	F	F	F
F	F	F	F	F

From the table, in last two columns, truth values of $p \vee (p \wedge q)$ and p are same for each choice of p and q. Hence

$$p \vee (p \wedge q) \equiv p$$

ii) $p \wedge (p \vee q) \equiv p$

Consider the table.

p	q	p ∨ q	p ∧ (p ∨ q)	p
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	F

From the table, last two columns are identical, hence

$$p \wedge (p \vee q) \equiv p$$

Note : Proofs of remaining identities are left as an exercise to the students.

Example 4.15 *$p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.*

Solution : Consider the table.

p	q	~ p	~ p ∨ q	p → q
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Hence $\sim p \vee q$ and $p \rightarrow q$ are logically equivalent.

Example 4.16 *$p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \land (q \rightarrow p)$ also $(p \rightarrow q) \land (q \rightarrow p)$ is logically equivalent to $(\sim p \lor q) \land (\sim q \lor p)$.*

Solution : Consider the table.

p	q	~ p	~ q	p → q	q → p	(p → q) ∧ (q → p)	p ↔ q	~ p ∨ q	~ q ∨ p	(~ p ∨ q) ∧ (~ q ∨ p)
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From the table,

$$p \leftrightarrow q \equiv (\sim p \lor q) \land (\sim q \lor p) \equiv (p \rightarrow q) \land (q \rightarrow p)$$

4.11 Normal Forms

In logic, with the help of truth table we can compare if two statements are equivalent.

But when more statements or propositions are involved, then this method is not practical.

[As if n propositions are involved, its truth values will be 2^n.]

Hence, it is necessary to apply another method.

One method is to transform S_1 and S_2 to some standard form S_1' and S_2'. Such that a simple comparison of S_1' and S_2' should establish whether $S_1' \equiv S_2'$.

The standard forms are called **normal forms** or **canonical forms.**

4.11.1 Disjunctive Normal Form [dnf]

Disjunctive normal form is a disjunction ($\lor$) of **fundamental conjunctions** ($\land$).

Now fundamental conjunctions ($\land$) are conjunction of simple statements [i.e. joining two statements by '$\land$'].

i.e. p, $p \land q$, $\sim p \land q$, $\sim p \land \sim q$, $p \land \sim p$, $q \land \sim q$, $p \land \sim q$, $\sim p$ are fundamental conjunctions.

Hence disjunction of fundamental conjunctions are joining fundamental conjunctions by '$\lor$'.

Examples

1) $(p \wedge q) \vee p \vee (q \wedge \sim p)$
2) $(p \wedge q \wedge r) \vee (p \wedge q' \wedge r) \vee (p' \wedge q \wedge r)$
3) $(p' \wedge q \wedge r) \vee (p \wedge q)$
4) $(p \wedge q) \vee (r)$
5) $(p \wedge q) \vee (r \wedge q) \vee (\sim r \wedge p)$

Remark

$q \wedge \sim q$, $p \wedge \sim p$ are always false. Hence if a fundamental conjunction contains at least one pair of (p and ~ p) or ($q \wedge \sim q$) etc., it will be false.

Example

1. Obtain the dnf of the form

$p \wedge (p \rightarrow q)$

$\equiv p \wedge (\sim p \vee q)$ [as $p \rightarrow q \equiv \sim p \vee q$]

$\equiv (p \wedge \sim p) \vee (p \wedge q)$ [Using distributive law]

$\equiv F \vee (p \wedge q)$

$\equiv (p \wedge q)$ **dnf**

4.11.2 Conjunctive Normal Form (cnf)

Conjunctive normal form is a **conjunction** ($\wedge$) of **fundamental disjunctions** ($\vee$).

Now fundamental disjunctions ($\vee$) are disjunction of simple statements.

[i.e. joining two statements by '$\vee$']

Example

$p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$, ~ p, ~ q, p, q, etc. are fundamental disjunctions.

Hence conjunction of fundamental disjunctions are joining fundamental disjunction by '$\wedge$'.

Examples

1. $(p \vee q) \wedge (q \vee r) \wedge (\sim p \vee \sim r)$
2. $p \wedge (\sim q \vee \sim r)$
3. $(p \vee q \vee \sim r) \wedge (p \vee q) \wedge (\sim p \vee q \vee \sim r)$

4.11.3 Truth Table Method to Find dnf

Let p be a proposition containing n variables $p_1, p_2, \ldots, p_i, \ldots, p_k, \ldots, p_n$.

To find its dnf from the truth table.

Step 1 : Consider the true values (T) from p.

Step 2 : Form the conjunction ('$\wedge$')

$(p_1 \wedge p_2 \wedge \ldots \wedge p_i \wedge \ldots \wedge p_k \wedge \ldots \wedge p_n)$,

where if p_i is true consider p_i and

if p_k is false consider $\sim p_k$

such a term is called minterm.

Step 3 : The disjunction of minterms is the dnf of the given form.

Conjunctive normal forms and disjunctive normal forms (cnf and dnf)

Example 4.17 *Obtain conjunctive normal form of*

i) $(\sim p \to r) \wedge (p \to q)$

ii) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

Solution : i) $(\sim p \to r) \wedge (p \to q)$

$\equiv (\sim(\sim p) \vee q) \wedge (\sim p \vee q)$ [As $p \to q \equiv \sim p \vee q$]

$\equiv (p \vee q) \wedge (\sim p \vee q)$ **cnf** [As $\sim\sim p \equiv p$]

ii) $(p \wedge q) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \vee (\sim p \wedge q \wedge r)) \wedge (q \vee (\sim p \wedge q \wedge r))$ [Using left distributive law]

$\equiv ((p \vee \sim p) \wedge (p \vee q) \wedge (p \vee r)) \wedge ((q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r))$

$\equiv (T \wedge (p \vee q) \wedge (p \vee r)) \wedge (q \vee \sim p) \wedge (q) \wedge (q \vee r)$

$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge (q) \wedge (q \vee r)$ **cnf**

Example 4.18 *Find conjunctive normal form and disjunctive normal form for the following without using truth table.*

$(p \to q) \wedge (q \to p)$

Solution : $(p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ **cnf**

$\equiv ((\sim p) \wedge (\sim q \vee p)) \vee (q \wedge (\sim q \vee p))$ [Using distributive law]

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge \sim q) \vee (q \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee F \vee F \vee (q \wedge p)$

$\equiv (\sim p \wedge \sim q) \vee (q \wedge p)$ **dnf**

Example 4.19 *Obtain disjunctive normal form of*

i) $(p \to q) \wedge (\sim p \wedge q)$

ii) $(p \wedge (p \to q)) \to q$

Solution : i) $(p \to q) \wedge (\sim p \wedge q)$

$\equiv (\sim p \vee q) \wedge (\sim p \wedge q)$ $[(p \to q) \equiv \sim p \vee q]$

$\equiv (\sim p \wedge (\sim p \wedge q) \vee (q \wedge \sim p \wedge q)$ [Using distributive law]

$\equiv ((\sim p \wedge \sim p) \wedge q) \vee (q \wedge q \wedge \sim p)$ [By associative and commutative laws]

$\equiv (\sim p \wedge q) \vee (q \wedge \sim p)$ **dnf** [By idempotent law]

ii) $(p \wedge (p \to q)) \to q$

$\equiv (p \wedge (\sim p \vee q)) \to q$ $[p \to q \equiv \sim p \vee q]$

$\equiv \sim (p \wedge (\sim p \vee q)) \vee q$

$\equiv \sim p \vee (\sim (\sim p \vee q)) \vee q$

$\equiv \sim p \vee (p \wedge \sim q) \vee q$ **dnf**

Example 4.20 *Obtain the conjunctive normal form of each of the following :*

i) $p \wedge (p \to q)$

ii) $\sim (p \vee q) \leftrightarrow (p \wedge q)$

Solution : i) $p \wedge (p \to q)$

$\equiv p \wedge (\sim p \vee q)$ **cnf**

ii) $\sim (p \vee q) \leftrightarrow (p \wedge q)$

$\equiv [\sim (\sim (p \vee q) \vee (p \wedge q)] \wedge [\sim (p \wedge q) \vee \sim (p \vee q)]$

$[p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)]$

$\equiv [(p \vee q) \vee (p \wedge q)] \wedge [(\sim p \vee \sim q) \vee (\sim p \wedge \sim q)]$

$\equiv [(p \vee q \vee p) \wedge (p \vee q \vee q)] \wedge [(\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)]$

$\equiv (p \vee q) \wedge (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$

$\equiv (p \vee q) \wedge (\sim p \vee \sim q)$ **cnf**

Example 4.21 *Eliminating conditional and biconditional, find logically equivalent forms of*

i) $(p \leftrightarrow (q \vee r)) \to \sim p$

ii) $((p \to q) \to q) \to p$

Solution : Using truth table method, find dnf, which is logically equivalent to given form.

p	q	r	$\sim p$	$q \vee r$	$p \leftrightarrow (q \vee r)$	$(p \leftrightarrow (q \vee r)) \rightarrow \sim p$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T →
F	T	T	T	T	F	T
F	T	F	T	T	F	T
F	F	T	T	T	F	T
F	F	F	T	F	T	T

Consider only (T) from last column and choose corresponding values (T) from p, q, r e.g. for marked row (→), corresponding p is true, q and r are false, so consider $(p \wedge q' \wedge r')$ or $(p \wedge \sim q \wedge \sim r)$.

Similarly for other (T) values choose truth values from column of p, q, r. Choose p if p is true otherwise p′ or ~ p if p is false.

Hence the logically equivalent form of

$$(p \leftrightarrow (q \vee r)) \rightarrow \sim p \equiv (p \wedge q' \wedge r') \vee (p' \wedge q \wedge r) \vee (p' \wedge q \wedge r') \vee (p' \wedge q' \wedge r) \vee (p' \wedge q' \wedge r')$$

Example 4.22 *Find dnf of*

$(p \rightarrow (q \wedge r)) \wedge (\sim p \rightarrow (\sim p \wedge \sim r))$ *by truth table method.*

Solution :

p	q	r	$\sim p$	$\sim r$	$q \wedge r$	$p \rightarrow q \wedge r$	$\sim p \wedge \sim r$	$\sim p \rightarrow (\sim p \wedge \sim r)$	$(p \rightarrow (q \wedge r))$ $(\wedge (\sim p \rightarrow (\sim p \wedge \sim r))$
T	T	T	F	F	T	T	F	T	T
T	F	T	F	F	F	F	F	T	F
T	T	F	F	T	F	F	F	T	F
T	F	F	F	T	F	F	F	T	F
F	T	T	T	F	T	T	F	F	F
F	F	T	T	F	F	T	F	F	F
F	T	F	T	T	F	T	T	T	T
F	F	F	T	T	F	T	T	T	T

Hence the logically equivalent form is

$(p \wedge q \wedge r) \vee (p' \wedge q \wedge r') \vee (p' \wedge q' \wedge r')$

5 Predicate Logic

Syllabus

Definition of Predicates; Statement functions, Variables, Quantifiers, Predicate Formulas, Free & Bound Variables; The Universe of Discourse, Examples, Valid Formulas & Equivalences, Examples.

Contents

5.1 Introduction

Let us consider

1) $x > 9$

2) "$x + y \leq 8$"

3) '$x + y + z = 9$'

These are sentences but not statements (or propositions) unless we know the values of x in 1, values of x, y in 2 and values of x, y, z in 3.

Such sentences are known as **'propositional functions'** or **'predicates'**.

Predicate

A **propositional function** or a **predicate** in one or more variables is a sentence that becomes a proposition, when variables are given definite values from the set of values it can take.

A predicate p contains n variables $x_1, x_2, \ldots x_n$ is called an n-place predicate.

1) '$x > 9$' is a one place predicate and is denoted by $p(x)$.

2) '$x + y \leq 8$' is two place predicate and is denoted by $p(x, y)$.

3) '$x + y + z = 9$' is three place predicate and is denoted by $p(x, y, z)$.

The values which the variables may assume constitute a collection called the **universe of discourse.**

Remark

1. A predicate is usually not a proposition but every proposition is a predicate (i.e. propositional function).
2. When specific values are given to the variables appearing in predicate, variables are bound. If all the variables are bound in a predicate then predicate becomes a proposition.

For example

i) $p(x) : x - 7 = 4$

Let the universe of discourse be the set of natural numbers.

Binding x by putting $x = 3$, we get false statement. Binding x by 11 we get true statement.

A second method of binding individual variables in a predicate is by quantification of the variable.

5.1.1 Universal Quantifier

If p(x) is a predicate with the variable x as an argument, the mathematical symbol for the quantifier 'for all', which is denoted by '∀', so that proposition can be written as,

"For all x, p(x)", which is interpreted as 'For all values of x, p(x) is true' is a proposition in which variable x is said to be **universally quantified** and '∀' is known as universal quantifier.

5.1.2 Existential Quantifiers

Suppose for the predicate p(x), ∀ x (p(x)) is false, but there exists atleast one value of x for which, p(x) is true, then we say that, x is bounded by **existential quantification.**

The word 'there exists' is denoted by '∃ '.

∃ x(p(x)) means 'there exists a value of x in the universe of discourse for which p(x) is true'.

Remark

Negation of ∀ x p(x) is ∀ x p(x) is not true or it is not the case that for all x, p(x) is true. This means atleast for some x p(x) is not true, in other words there exists an x such that ~ p(x) is true.

Hence $\forall x\,[p(x)] \equiv \exists x\,[-p(x)]$

Example 5.1 *Determine the validity of following argument.*

s_1 : *All my friends are musicians.*

s_2 : *John is my friend.*

s_3 :*None of my neighbours are musiscians.*

---

s : John is not my neighbour.

Solution : Let p : All my friends are musician.

q : John is my friend.

r : My neighbours are musician.

$s_1 : p$

$s_2 : q$

$s_3 : \sim r$

Hence $s_1 \wedge s_2 \wedge s_3$,

i.e. $p \wedge q \wedge \sim r$ is statement form.

Conclusion : John is not my neighbour.

Consider t_1 : John is musician.

t_2 : John is my neighbour.

Then from the premises

$$q \wedge p \rightarrow t_1$$

As John is my friend and all my friends are musician ⇒ John is musician.

Now $p \wedge q \wedge \sim r \rightarrow t_1 \wedge \sim r$ is true.

i.e. John is musician and my neighbours are not musician. Hence John is not my neighbour. Therefore given argument is valid.

Example 5.2 *Determine the validity of argument given :*

s_1 : *If I like mathematics then I will study.*

s_2 :*EitherI will study or I wil fail.*

s : *If I fail then I do not like mathematics.*

Solution : Let p : I like mathematics.

q : I will study.

r : I will fail.

$s_1 : p \rightarrow q$

$s_2 : q \vee r$

$s : r \rightarrow \sim p$

So

$$\begin{array}{l} p \rightarrow q \\ \underline{q \vee r} \\ r \rightarrow \sim p \end{array}$$

Using truth table

p	q	r	$p \rightarrow q$	$q \vee r$	$\sim p$	$r \rightarrow \sim p$
T	T	T	T	T	F	F

Hence the argument is not valid.

As corresponding to truth values T, T, T of p, q, r, s_1 is T, s_2 is T but s is false.

Example 5.3 *Over the universe of book defined propositions*

B(x) : x has blue cover.

M(x) : x is maths book.

I(x) : x published in India.

Translate the following.

i) $\forall x(M(x) \wedge I(x) \rightarrow B(x))$

ii) There are maths books published outside India.

Solution : i) All mathematics book published in India have blue cover.

ii) $\exists x\,(M(x) \wedge \sim I(x))$

Example 5.4 *Rewrite the following statements using quantifiers variables and predicate symbols.*

i) All birds can fly.

ii) Not all birds can fly.

iii) Some men are genius.

iv) Some numbers are not rational.

v) There is a student who likes mathematics but not Geography.

vi) Each integer is either even or odd.

Solution : i) $p(x)$: x is a bird.

$q(x)$: x can fly.

Then statement can be written as,

$$\forall x\,[p(x) \rightarrow q(x)]$$

ii) $\exists x\,[p(x) \wedge \sim q(x)]$ or equivalently $\sim [\forall x(p(x) \rightarrow q(x))]$

iii) $A(x)$: x is a man.

$B(x)$: x is a genious.

Then the statement can be written as,

$$\exists x\,[A(x) \wedge B(x)]$$

iv) $N(x)$: x is a number.

$Q(x)$: x is rational.

Then the statement can be written as $\exists x\,[N(x) \wedge \sim Q(x)]$ or equivalently $\sim [\forall x\,(N(x) \rightarrow Q(x))]$.

v) $S(x)$: x is a student.

$M(x)$: x likes mathematics.

$G(x)$: x likes Geography.

Then the statement in symbolic form is

$\exists x\,[S(x) \wedge M(x) \wedge \sim G(x)]$

vi) $I(x)$: x is an integer.

$E(x)$: x is even.

$O(x)$: x is odd.

Then the statement in symbolic form is $\forall x\,[I(x) \rightarrow E(x) \vee O(x)]$

Example 5.5 *State whether true or false.*

i) $(\forall n \in N), (n+4>3)$,

ii) $(\forall n \in N), (n+2>8)$,

iii) $(\exists n \in N), (n+4<7)$,

iv) $(\exists n \in N), (n+6<4)$

Solution : i) The statement is true.

As for all natural numbers n+4 is always greater than 3.

ii) The statement is false.

As say n = 3

then 3 + 2 < 8.

iii) The statement is true.

As there exist natural numbers say n = 1, 2 such that 1 + 4 < 7

Also 2 + 4 < 7

iv) The statement is false.

As the smallest natural number is 1 and 1 + 6 > 4.

Example 5.6 *Negate each of the statement.*

i) $\forall x, |x| = x$

ii) $\exists x, x^2 = x$

iii) *If there is a riot, then someone is killed.*

iv) *It is day light and all the people are arisen.*

Solution : i) $\exists x, |x| \neq x$.

ii) $\forall x, x^2 \neq x$

iii) p : There is a riot.

q : Someone is killed.

Given statement is $p \rightarrow q$

Hence $\sim (p \rightarrow q)$

$\equiv \sim (\sim p \vee q)$

$\equiv p \wedge \sim q$

$\equiv$ There is a riot and someone is not killed.

iv) p : It is a day light.

q : All the people are arisen.

Given statement is $p \wedge q$.

Hence $\sim (p \wedge q) = \sim p \vee \sim q$.

Hence either it is not a day light or all the people are not arisen.

Example 5.7 *Determine the truth value of each of the statement and negate every statement.*

i) $\exists x, x + 2 = x$

ii) $\forall x, x + 1 > x$

Solution : i) False.

Negation of $\exists x, x + 2 = x$

is $\forall x, x + 2 \neq x$

ii) True.

Negation of $\forall x, x + 1 > x$

is $\exists x, x + 1 \leq x$.

Predicate Formulas :

Let $P(x_1, x_2, \ldots x_n)$ denote an n-place predicate formula in which the letter P is an n-place predicate and $x_1, x_2, \ldots x_n$ are identical variables.

In general $P(x_1, x_2, \ldots x_n)$ will be known as atomic formula of the statement.

A well formed formula of predicate calculus is obtained by using the following rules.

1. An atomic formula is a well-formed formula.
2. If A is a well formed formula then also $\sim A$ is a well-formed formula.
3. If A and B are well formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, $(A \leftrightarrow B)$ are also well-formed formulas.
4. If A is a well-formed formula and x is any variable, then $(x)A$ and $\exists x$ are well-formed formulas.
5. Only those formulas which are obtained by using rule 1 to 4 are well-formed formulas.

Free and Bound Variables

When a quantifier is used in the variable x or when we assign a value to this variable, then the occurrence of the variable is **bound**. An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.

Examples :

1) $(x)\ P(x, y)$

$P(x, y)$ is the scope of the quantifier. Both occurrence of x are bound occurrence while the occurrence of y is a free occurrence.

2) $(x)\ P(x) \rightarrow (\exists y)\ R(x, y)$

Both x and y are bound occurrence.

Equivalences

Let A and B be any two predicate formulas defined over a common universe denoted by the symbol E. If, for every assignment of object names from the universe of discourse E to each of the variables appearing in A and B, the resulting statement have the same truth values, then the predicate formulas A and B are said to be equivalent to each over E.

If E is arbitrary, then A and B are equivalent,

i.e. $A \Leftrightarrow B$

Four Types of Equivalences

"all true" $\{(\forall x),(P(x))\} \Leftrightarrow \{\sim(\exists x)(\sim P(x))\}$ "none false"

"all false" $\{(\forall x),(P(x))\} \Leftrightarrow \{\sim(\exists x)(P(x))\}$ "none true"

"not all true" $\{\sim(\forall x),(P(x))\} \Leftrightarrow \{(\exists x)(\sim P(x))\}$ "atleast one false"

"not all false" $\{\sim(\forall x),(\sim P(x))\} \Leftrightarrow \{(\exists x)(P(x))\}$ "atleast one true"

Valid Formulas

A formula A is said to be valid in E, if for every assignment of object names from E to the corresponding variables in A and for every assignment of statements to statement variables, the resulting statement have truth value T.

Example 5.8 $(\forall x)\ (P(x)) \Rightarrow (\exists x)\ (P(x))$ *is a logically valid statement.*

Solution : If $\forall x$, P(x) is true in some particular universe, then the universe has at least one object a in it and P(b) is a true statement for every b in the universe. In particular P(a) must be true.

So, $(\exists x)\ (P(x))$ is valid formula.

□□□

6 Relations

Syllabus

Definition, Binary Relation, Representation, Domain, Range, Universal Relation, Void Relation, Union, Intersection, and Complement Operations on Relations, Properties of Binary Relations in a Set : Reflexive, Symmetric, Transitive, Anti-symmetric Relations, Relation Matrix and Graph of a Relation; Partition and Covering of a Set, Equivalence Relation, Equivalence Classes, Compatibility Relation, Maximum Compatibility Block, Composite Relation, Converse of a Relation, Transitive Closure of a Relation R in Set X.

Contents

6.1 Ordered Pair

Let A and B be two sets. Let $a \in A$ and $b \in B$. Then (a, b) denotes **ordered pair** and a is known as first co-ordinate of the ordered pair (a, b) and b as second co-ordinate of the ordered pair (a, b).

6.1.1 Cartesian Product

Let A and B be two sets then $A \times B$ is called Cartesian product of A and B.

$A \times B$ is a set of all distinct ordered pairs, in which first co-ordinate of ordered pair is from set A and second co-ordinate is from set B.

$$A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$$

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

then $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$

$$= \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$

$$(1, 2) \notin A \times B$$

as $2 \notin B$

$$(a, b) \notin A \times B$$

as $a \notin A$

$$(b, 3) \notin A \times B$$

as $b \notin A$ and $3 \notin B$

$\Rightarrow$ $(a, b) \notin A \times B$, either $a \notin A$ or $b \notin B$ or both.

Hence $A \times B \neq B \times A$

So, Cartesian product is non commutative, if A and B are non empty and different sets. i.e. $A \neq \varnothing$, $B \neq \varnothing$ and $A \neq B$.

Similarly $A \times B \times C$ is a set of all distinct ordered triple.

$$A \times B \times C = \{(a, b, c) : a \in A \text{ and } b \in B \text{ and } c \in C\}$$

this way,

$$A_1 \times A_2 \times A_3 \ldots \times A_n = \{(a_1, a_2, a_3, \ldots a_n) : a_1 \in A_1, a_2 \in A_2, \ldots a_i \in A_i, \ldots a_n \in A_n\}$$

is set of ordered n-tuples.

Remarks :

1. If A contains 3 elements and B contains 4 elements then $A \times B$ contains $3 \times 4 = 12$ elements.

$$\text{i.e. } |A \times B| = m \cdot n$$
$$\text{if } |A| = m \text{ and } |B| = n$$

2. $A \times B \times C = \{(a, b, c) ; a \in A, b \in B \text{ and } c \in C\}$

$$A \times (B \times C) = \{(a, (b, c)) ; a \in A, (b, c) \in B \times C\}$$
$$(A \times B) \times C = \{((a, b), c) ; (a, b) \in A \times B, c \in C\}$$

All these $A \times B \times C$, $(A \times B) \times C$, $A \times (B \times C)$ are different.

Normally product of A, B, C, we define as $A \times B \times C$.

Theorem :

If A, B, C are sets then,

i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

iii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

iv) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof :

Let $(a, b) \in A \times (B \cap C)$

$$\Leftrightarrow a \in A \text{ and } b \in (B \cap C)$$
$$\Leftrightarrow a \in A \text{ and } (b \in B \text{ and } b \in C)$$
$$\Leftrightarrow (a \in A \text{ and } b \in B) \text{ and } (a \in A \text{ and } b \in C)$$
$$\Leftrightarrow (a, b) \in (A \times B) \text{ and } (a, b) \in (A \times C)$$
$$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$$

$$\Rightarrow A \times (B \cap C) = (A \times B) \cap (A \times C)$$

Students are advised to prove the other three results, as an exercise.

Remarks :

1. If $\mathfrak{R}$ is the set of real numbers then $\mathfrak{R} \times \mathfrak{R}$ we represent by $\mathfrak{R}^2$ and $\mathfrak{R} \times \mathfrak{R} \times \mathfrak{R}$ by $\mathfrak{R}^3$.
2. $\mathfrak{R} \times \mathfrak{R} = \mathfrak{R}^2$ is called the **Euclidean plane**.

6.2 Relation or Binary Relation

Let S be a set of family members

$$S = \{f, m, b_1, b_2, s_1\}$$

Number of elements of S = 5 [also known as cardinality of S and denoted by n (S) or $|S|$] f $\rightarrow$ father, m $\rightarrow$ mother , $b_1, b_2 \rightarrow$ brothers, $s_1 \rightarrow$ sister then $S \times S$, the carterian product is a set of ordered pair.

$$S \times S = \begin{bmatrix} (f,f) & (f,m) & (f,b_1) & (f,b_2) & (f,s_1) \\ (m,f) & (m,m) & (m,b_1) & (m,b_2) & (m,s_1) \\ (b_1,f) & (b_1,m) & (b_1,b_1) & (b_1,b_2) & (b_1,s_1) \\ (b_2,f) & (b_2,m) & (b_2,b_1) & (b_2,b_2) & (b_2,s_1) \\ (s_1,f) & (s_1,m) & (s_1,b_1) & (s_1,b_2) & (s_1,s_1) \end{bmatrix}$$

Number of elements in $S \times S$ is $5 \times 5 = 25$ i.e. cardinality of $S \times S$ is 25.

Now let R be a relation on the set S say R be father relation.

then $R \subset A \times B$

$R = \{(f, b_1), (f, b_2), (f, s_1)\}$

f is father of b_1, denoted by f R b or $(f, b) \in R$.

f is father of b_2, f is father of s_1

i.e. $R \subset S \times S$

Remarks :

1. Relation is always defined on set.
2. Binary relation is a subset of cartesian product of two sets. Both sets may be same or different. If R is defined from set A to set B then,

   $R \subset A \times B$
3. The sets A and B are also known as set of departure and the set of destination of R respectively.

Examples :

1. A is the set of natural number N.

   The set $\{(x, y) : x = y\}$ means that x and y are so related that x = y.

then $R = \{(1, 1), (2, 2), (3, 3) \ldots, (n, n) \ldots\ldots\}$

2. $A = N$

$R = \{(x, y) : y = x^2\}$

then $R = \{(1,1), (2,4), (3,9), (4,16), (5,25) \ldots (n, n^2) \ldots\}$

3. $A = \{1, 2, 3, \ldots\ldots 19, 20\}$

$R = \{(x, y) : x = 3y\}$

then $R = \{(3, 1), (6, 2), (9, 3), (12, 4), (15, 5), (18, 6)\}$

4. $A = \{2, 3, 5, 6\}$

$R = \{(x, y) : x | y \ (x \text{ divides } y)\}$

then $R = \{(2,2)\ (2,6), (3,3), (3,6), (5,5), (6,6)\}$

Result - 1 :

If A, B, C are three sets and $A \subseteq B$, then

$$A \times C \subseteq B \times C$$

Proof :

Let $(x, y) \in A \times C \Rightarrow x \in A$ and $y \in C$

$\Rightarrow x \in B$ and $y \in C$ (as $A \subseteq B$)

$\Rightarrow (x, y) \in B \times C$

$\Rightarrow A \times C \subseteq B \times C$

Result - 2 :

$$(A - B) \times C = (A \times C) - (B \times C)$$

Let $(x, y) \in (A - B) \times C$

$\Leftrightarrow x \in (A - B)$ and $y \in C$

$\Leftrightarrow (x \in A$ and $x \notin B)$ and $y \in C$

$\Leftrightarrow (x \in A$ and $y \in C)$ and $(x \notin B$ and $y \in C)$

$\Leftrightarrow (x, y) \in A \times C$ and $(x, y) \notin (B \times C)$

$\Leftrightarrow (x, y) \in (A \times C) - (B \times C)$

$\Rightarrow (A - B) \times C = (A \times C) - (B \times C).$

6.2.1 Domain and Range of a Relation

Let A and B be two sets and R be a relation from A to B

i.e. $R \subset A \times B$

and $R = \{(a, b) : a \in A \text{ and } b \in B\}$

The set of all first co-ordinates of R is known as domain of R and the set of all second co-ordinates of R is known as range of R.

Thus

$$\text{domain of } R = \{a : (a, b) \in R\}$$

$$\text{and range of } R = \{b : (a, b) \in R\}$$

6.2.2 Matrix Representation of a Relation

Let $A = \{a_1, a_2 \ldots. a_n\}$

and $B = \{b_1, b_2 \ldots. b_m\}$

A and B are finite sets containing n and m elements respectively. Then $A \times B$ will contain $n \times m$ elements.

Let R be a relation from A to B

then $R \subseteq A \times B$

To represent R as matrix $n \times m$ known as M_R [matrix of relation] = $[m_{ij}]$

We take

$$m_{ij} = 1 \text{ if } a_i R b_j \text{ or } (a_i, b_j) \in R$$

$$= 0 \text{ if } a_i \not{R} b_j \text{ or } (a_i, b_j) \notin R.$$

Example 6.1 $A = \{1, 2, 3, 4\}$

$B = \{1, 4, 6, 8, 9\}$

a R b if and only if $b = a^2$ then find the relation matrix M_R.

Solution :

$$R = \{(1, 1), (2, 4), (3, 9)\}$$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 4 & 6 & 8 & 9 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{4 \times 5} \end{array}$$

Example 6.2 $A = \{1, 2, 3, 4, 6\} = B$,

aRb if and only if $a \mid b$ then find the relation matrix M_R.

Solution :

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 4), (2, 6), (3, 6), (2, 2), (3, 3), (4, 4), (6, 6)\}$$

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{5\times5} \end{array}$$

Example 6.3 *$A = \{1, 2, 3, 5, 6, 10, 15, 30\} = B$, aRb iff $a|b$. Find relation matrix.*

Solution :

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 5 \\ 6 \\ 10 \\ 15 \\ 30 \end{array}\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 5 & 6 & 10 & 15 & 30 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{8\times8} \end{array}$$

6.2.3 Relation Matrix Operations

Relation matrix is known as Boolean matrix as the entries are either 0 or 1.

Let $\quad A = [a_{ij}]$ and $B = [b_{ij}]$ be $m \times n$

then $\quad A + B = [a_{ij} + b_{ij}]$

$\quad\quad = [c_{ij}]$

Where $\quad c_{ij} = 1$ if $a_{ij} = 1$ or $b_{ij} = 1$

and $\quad c_{ij} = 0$ if $a_{ij} = 0$ and $b_{ij} = 0$

also

If $\quad A = [a_{ij}]_{m\times n}$, $B = [b_{jk}]_{n\times k}$

Boolean martix then,

$$A \cdot B = [a_{ij} b_{jk}]$$

$$= [d_{ik}]$$

and $\quad d_{ik} = 1$ if $a_{ij} = 1$ and $b_{jk} = 1$

$\quad\quad = 0$ if $a_{ij} = 0$ or $b_{jk} = 0$

e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

6.2.4 Properties of Relation Matrix

Let R_1 be a relation from A to B and R_2 be a relation from B to C. Then the relation matrices satisfy the following properties,

$$M_{R1 \cdot R2} = M_{R1} \cdot M_{R_2}$$

$$M_{R^{-1}} = \text{Transpose of } M_R$$

$$M_{(R1 \cdot R2)}^{-1} = M_{R_2^{-1}} \cdot M_{R_1^{-1}}$$

6.2.5 Graphical Representation of a Relation (Digraph)

Let A be a finite set and R be a relation defined on R, then R can be represented by graph.

The elements of set A are represented by circles or points. These elements are known as **nodes** or **vertices**. If aRa then, this can be represented by loop around a. If aRb then, this can be shown by an arc from a to b with an arrow from a to b.

i.e. a $\rightarrow$ b. These arcs and loops are known as edges of the graph.

This graphical representation is known as **directed graphs** or **digraph of R.**

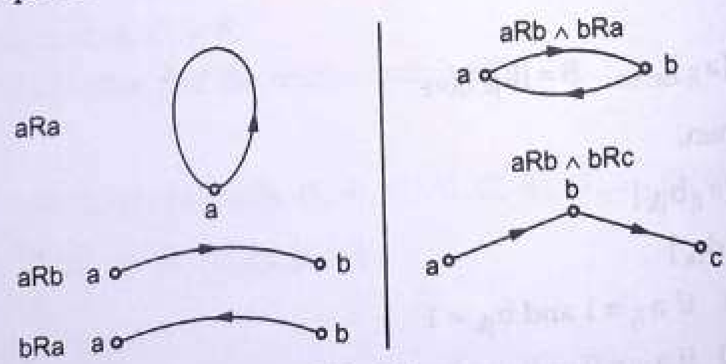


Fig. 6.1

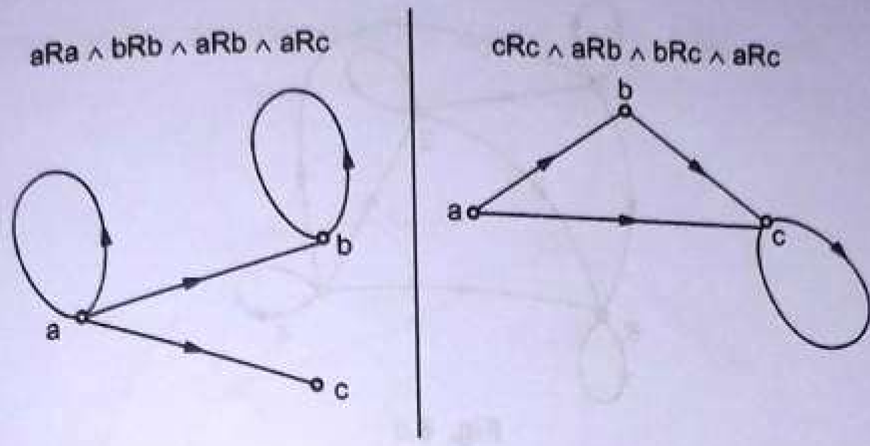


Fig. 6.2

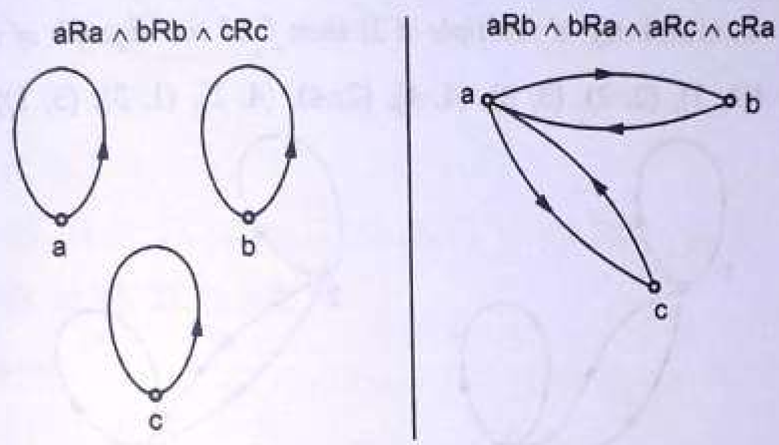


Fig. 6.3

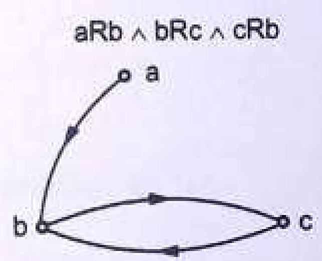


Fig. 6.4

Example 6.4 *A = {1, 2, 4, 8}*

aRb iff a | b *(a divides b) then find the digraph of relation.*

Solution :

R = {(1, 1), (1, 2), (1, 4), (1, 8), (2, 2), (2, 4), (2, 8), (4, 4), (4, 8), (8, 8)}

Hence digraph of R is the following

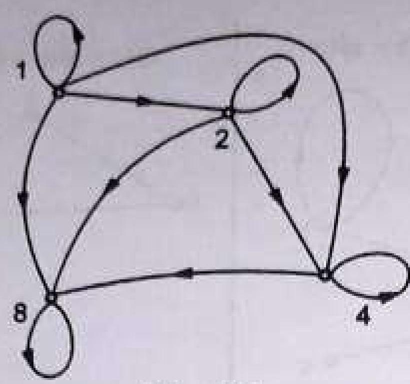


Fig. 6.5

Example 6.5 $A = \{1, 2, 3, 4\}$

If $R = \{(a,b) \mid (a-b) \text{ is an integral multiple of } 2\}$ *then find the digraph of relation.*

Solution : $R = \{(1, 1), (2, 2), (3, 3), (4, 4), (2, 4), (4, 2), (1, 3), (3, 1)\}$

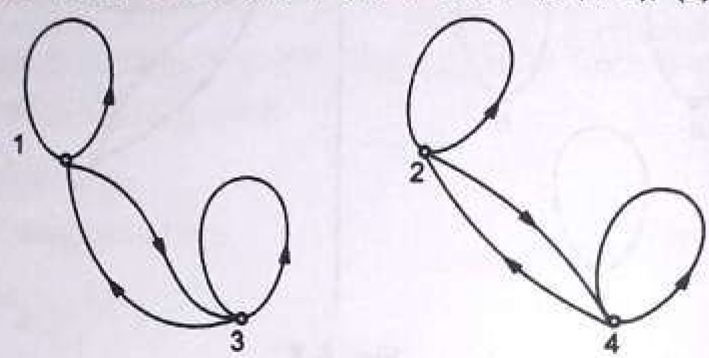


Fig. 6.6

Example 6.6 $A = \{1, 2, 3, 4\}$

and $M_R = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Draw the digraph of R.

Solution : $R = \{(1, 1), (1, 4), (2, 2), (2, 4), (3, 1), (3, 2), (4, 1)\}$

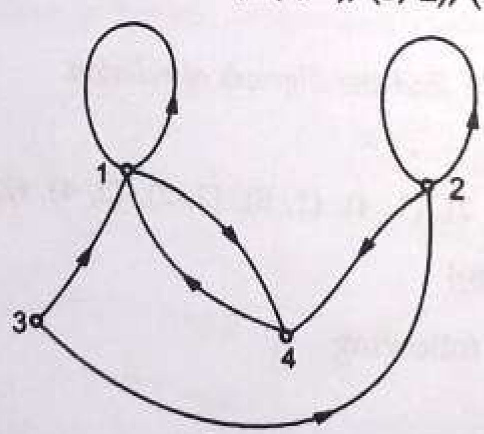


Fig. 6.7

Example 6.7 *Find the relation determined by the digraph and give its matrix.*

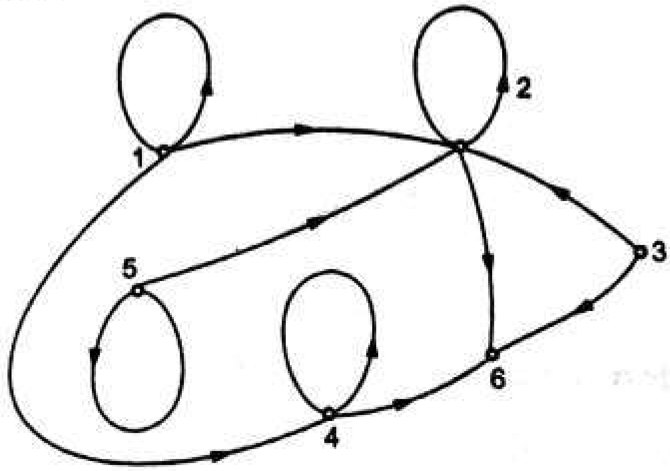


Fig. 6.8

Solution : A = {1, 2, 3, 4, 5, 6,}

R = {(1, 1), (1, 2), (2, 2), (1, 4), (3, 2), (3, 6), (4, 6), (4, 4), (5, 5), (5, 2), (2, 6)}

Find the relation matrix.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 6.8 *A = {1, 2, 3, 4} the relation martix is*

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Draw its digraph.

Solution : R = {(1, 1), (1, 3), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3)}

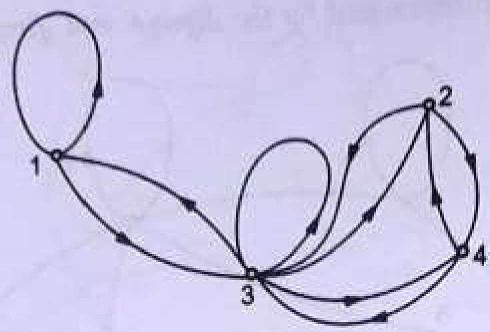


Fig. 6.9

6.3 Inverse Relation (or Converse Relation)

If R is a relation from A to B then inverse relation of R, denoted by R^{-1} is a relation from B to A

$$R^{-1} = \{(y, x) : y \in B, x \in A \text{ and } (x, y) \in R\}$$

If $(x, y) \in R$, i.e. $x \, R \, y$

then $y R^{-1} x$ or $(y, x) \in R^{-1}$

The inverse relation is also known as converse relation and hence sometimes it is denoted by R^c.

Example :

1. If R is father relation on set S then

$$R = \{(f, b_1), (f, b_2), (f, s_1)\}$$

then inverse relation of R can be 'son relation' or 'daughter relation' so,

$$R_1^{-1} = \{(b_1, f), (b_2, f)\} \text{ for relation son}$$

and $$R_2^{-1} = \{(s_1, f)\} \text{ for daughter relation.}$$

2. If $A = \{a_1, a_2, a_3\}$, $B = \{1, 2\}$ and relation R is defined from A to B

and $$R = \{(a_1, 1), (a_2, 2), (a_3, 2)\}$$

then relation R^{-1} is defined from B to A

and $$R^{-1} = \{(1, a_1), (2, a_2), (2, a_3)\}$$

Results :

If R_1, R_2 and R are relations from A to B then

i) $(R^{-1})^{-1} = R$

ii) $(R_1 \cup R_2)^{-1} = R_2^{-1} \cup R_1^{-1}$

iii) $(R_1 \cap R_2)^{-1} = R_2^{-1} \cap R_1^{-1}$

6.4 Complement of a Relation

Let A and B be two non-empty sets. A relation R from A to B is defined as

$$R = \{(a,b) : a \in A \text{ and } b \in B\}$$

then R′ (complement of R) is defined as

$$R' = \{(a,b) : (a,b) \notin R\}$$

i.e. $aR'b$ iff $a\not{R}b$.

Results :

$$\left.\begin{aligned} &1)\quad (R_1 \cup R_2)' = R_2' \cap R_1' \\ &2)\quad (R_1 \cap R_2)' = R_2' \cup R_1' \end{aligned}\right\} \text{De Morgan's laws}$$

If relation is defined between two elements i.e. $R \subset A \times B$ then it is called binary relation. If relation is defined on set A for one element then R is known as unary relation. e.g. negation of an element is a unary relation.

If relation is defined between three element

i.e. $R \subset A \times B \times C$ then relation is called ternary relation.

Remark :

If $R \subset A \times B$

then $R^{-1} \subset B \times A$

6.5 Composite of Binary Relations

Definition

Let R_1 be a relation from A to B and R_2 a relation from B to C. The composite relation from A to C, denoted by $R_1 \cdot R_2$(or $R_1 R_2$) is defined as

$$R_1 \cdot R_2 = \{(a, c) \mid a \in A \text{ and } c \in C \text{ and } b \in B \text{ such that } (a, b) \in R_1 \text{ and } (b, c) \in R_2\}$$

Example :

1. $A = \{1, 2, 3, 4, 5\}$

$R_1 = \{(1, 2), (1, 3), (2, 4), (3, 5)\}$

$$R_2 = \{(3, 2), (4, 5), (5, 1)\}$$

then $R_1 \cdot R_2 = \{(1, 2), (2, 5), (3, 1)\}$

also $R_2 \cdot R_1 = \{(3, 4), (5, 2), (5, 3)\}$

clearly $R_1 \cdot R_2 \neq R_2 \cdot R_1$

6.6 Relation in a Set

Let R be a relation from A to itself, then R is a relation in A.

i.e. $R \subset A \times A$.

6.6.1 Identity Relation in a Set

Let A be a set. The relation I_A defined by, $I_A = \{(a, b) : a \in A, b \in A, a = b\}$ is called the identity relation. Thus identity relation in a set A is the set of ordered pair (a, b) of $A \times A$ for which a = b.

Example :

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$

$$I_A = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7)\}$$

6.6.2 Universal Relation in a Set

Let A be any set and R be any relation on set A, then $R \subseteq A \times A$.

If $R = A \times A$, then the relation R is known as universal relation.

6.6.3 Void Relation (Empty Relation)

A relation R on a set A is called void relation. If no element of set A is related to any element of A. Hence $R = \phi$

6.7 Types of Relation

6.7.1 Reflexive Relation

Let R be a relation in a set A, i.e. $R \subset A \times A$ then R is known as reflexive relation if

$\forall a \in A$

$(a, a) \in R$.

i.e. every element is relataed with itself.

Example 1

If A is a set of all straight lines in 2-D plane and R is a relation.

$$R = \{(a, b) : \text{a is parallel to b}\}$$

then R is reflexive relation as every straight line is parallel to itself.

Remarks :

1. If R is reflexive on set A. i.e. $\forall$ a $\in$ A, a R a.

   i) In matrix of relation (M_R) all the diagonal elements will be 1.

   ii) Digraph of reflexive relation will have loop for every element of A.

2. If R is not reflexive, then R is known as **irreflexive** if $\forall$ a $\in$ A

   $(a, a) \notin R$ or $a \not R a$.

   hence all the diagonal entries in M_R will be zero and diagraph of **irreflexive** relation will have no loop for any element of A.

Example 2

I is the set of integers and R is a relation.

$$R = \{(a, b) : a \mid b \text{ (a divides b)}\}$$

then R is reflexive relation as every element divide itself.

Example 3

If I is a set of integers and R is a relation

$$R = \{(a, b) ;\ 5 \mid a - b \ \ (5 \text{ divides} (a-b))\}$$

then R is reflexive relation as $[5 \mid a - a \text{ or } 5 \mid 0]$ zero is divisible by every non zero number.

Example 4

If A is set of all straight lines in 2-D plane and R is a relation.

$$R = \{(a, b) : \text{a is perpendicular to b}\}$$

then R is **not** reflexive relation as no straight line is perpendicular to itself. This is an example of irreflexive relation.

6.7.2 Symmetric Relation

Let R be a relation in a set A i.e. $R \subset A \times A$. Then the relation R is said to be symmetric relation

if $\quad (a, b) \in R \Rightarrow (b, a) \in R$

i.e. if a is related with b with the relation R then b should be related with a with the same relation R.

$$aRb \Rightarrow bRa$$

in other way $R = R^{-1}$ then relation is symmetric.

Example 1

A is a set of all straight lines in 2 - D plane and R is a relation.

$$R = \{(a, b) \ ; \ a \text{ is parallel to } b\}$$

then R is symmetric relation as if a is parallel to b then b is also parallel to a.

i.e. $(a, b) \in R \Rightarrow (b, a) \in R$

Example 2

A is a set of all straight lines in 2 - D plane and R is a relation.

$$R = \{(a, b) \ ; \ a \text{ is perpendicular to } b\}$$

then R is symmetric relation as if a is perpendicular to b then b is also perpendicular to a.

Digraph of a symmetric relation

$$aRb \Rightarrow bRa$$

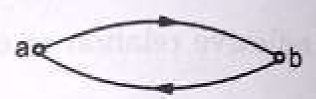


Fig. 6.10

Remarks :

Matrix of a relation M_R will have same entries for ij^{th} value and ji^{th} value

i.e. if $a_{ij} = 1$, then $a_{ji} = 1$

also if $a_{ij} = 0$, then $a_{ji} = 0$

in short transpose of relation matrix $(M_R)'$ is same as relation matrix (M_R).

i.e. $(M_R)' = M_R$

Also $R = R^{-1}$, then R is symmetric relation.

6.7.2.1 Compatible Relation

A relation R on a set S is said to be compatible if it is **reflexive** and **symmetric.**

Remark :

If R is a compatible relation its relation matrix is symmetric with the diagonal elements being 1.

6.7.3 Antisymmetric Relation

Let R be a relation in a set A i.e. $R \subset A \times A$. The relation R is said to be antisymmetric relation. If $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$.

i.e. if a is related with b with relation R and also b is related with a with the same relation R then a and b are same.

Example 1

$A \subseteq B,\ B \subseteq A \ \Rightarrow\ A = B$

So 'subset' relation is antisymmetric relation.

Example 2

$a \leq b,\ b \leq a \ \Rightarrow\ a = b$

So 'less than or equal to' relation is antisymmetric relation.

Example 3

As example 2

$a \geq b, b \geq a \ \Rightarrow\ a = b$

'greater than or equal to' relation is antisymmetric relation.

Example 4

N is a set of natural numbers.

$$R = \{(a, b) : a \mid b, \text{a divides b}\}$$

then R is antisymmetric relation.

i.e. if $a \mid b$ and $b \mid a \Rightarrow a = b$

But the same relation is not antisymmetric on the set of integers as $-4 \mid 4$ also $4 \mid -4$ but $4 \neq -4$.

6.7.4 Transitive Relation

Let R be a relation on set A i.e. $R \subset A \times A$, then R is called transitive relation on set A if

$$(a, b) \in R,\ (b, c) \in R \Rightarrow (a, c) \in R$$

i.e. if a is related with b and b is related with c and this shows a is related with c with the same relation then R is called transitive relation.

i.e.

$$aRb,\ bRc \Rightarrow aRc$$

Example 1

$$A \subseteq B,\ B \subseteq C \Rightarrow A \subseteq C$$

So 'subset' relation is transitive relation.

Example 2

a is parallel to b, b is parallel to c then a is parallel to c.

'Parallel' relation is transitive on the set of straight line in 2-D plane.

Example 3

Let I be set of integers and R is a relation defined on set I.

$$R = \{(a, b)\ ;\ a \mid b\}$$

then R is transitive relation as if $a \mid b$ and $b \mid c \Rightarrow a \mid c$

i.e. $(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R$

or $aRb, bRc \Rightarrow aRc$

Example 4

R is a relation on the set of integers I.

$$R = \{(a, b) \text{ iff } 5 \mid a - b\}$$

$$aRb \Leftrightarrow 5 \mid a - b$$

then R is transitive relation as aRb, bRc

$$\Rightarrow 5 \mid a - b \text{ and } 5 \mid b - c$$

$$\Rightarrow 5 \mid (a - b + b - c)$$

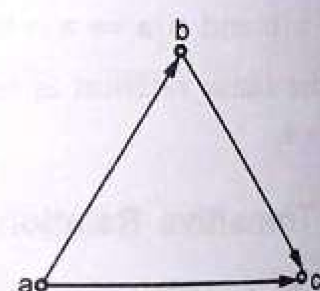
$$\Rightarrow 5 \mid a - c$$

$$\Rightarrow a R c$$

hence $aRb, bRc \Rightarrow aRc$

So R is transitive relation.

Digraph of transitive relation



$aRb, bRc \Rightarrow aRc$

Fig. 6.11

6.7.5 Transitive Closure

Definition : The transitive closure of a relation R is the smallest transitive relation containing R.

Transitive closure of R is denoted by R^*.

To find the transitive closure of R.

Let A be a set and cardinality of A is n,

i.e. $|A| = n$

and let R be a relation on A. Then,

$$R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^n$$

where $R^2 = R \cdot R$

$R^3 = R \cdot R \cdot R$ etc.

Example 6.9 *If A = {1, 2, 3, 4, 5}*

and R = {(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)}

find its transitive closure.

Solution : Let R^* the transitive closure of R

where $R = \{(1, 2), (3, 4), (4, 5), (4, 1), (1, 1)\}$

then $R^* = R \cup R^2 \cup R^3 \cup \ldots \cup R^k \cup \ldots$

$$R \cdot R = R^2 = \{(3, 5), (3, 1), (4, 2), (4, 1), (1, 1), (1, 2)\}$$

$$R \cdot (R \cdot R) = R^3 = \{(3, 2), (3, 1), (4, 1), (4, 2), (1, 1), (1, 2)\}$$

$$R^4 = \{(3, 1), (3, 2), (4, 1), (4, 2), (1, 1), (1, 2)\}$$

$$R^5 = \{(3, 1), (3, 2), (4, 1), (4, 2), (1, 1), (1, 2)\}$$

Hence $R_3 = R_4 = R_5$

So $R^* = R \cup R^2 \cup R^3$

$$R^* = \{(1, 2), (3, 4), (4, 5), (4, 1), (1, 1), (3, 5), (3, 1), (4, 2), (3, 2)\}$$

6.7.5.1 Warshall's Algorithm

To find the transitive closure of a relation R, sometime the method of computing R^2, R^3, ... is inefficient for large number of set. Warshall's algorithm is an efficient method for finding transitive closure of a relation R.

Let R be a relation on a set

$$A = \{a_1, a_2, \ldots a_n\}$$

If x, $x_1, x_2 \ldots x_n$, y is a path in R then the vertices other than x, y are known as **interior vertices** and xRx_1, x_1Rx_2, ... $x_nR\,y$.

$x_1, x_2, \ldots x_n$ are interior vertices of the path. For $1 \le i \le n$, define a boolean matrix W_k has 1 in position i, j if and only if there is a path from a_i to a_j in R whose interior vertices, if any are from the set $\{a_1, a_2, \ldots a_k\}$.

Since any vertex must come from the set $\{a_1, a_2, \ldots a_n\}$, it follows that the matrix W_n has a 1 in position i, j if and only if some path in R connects a_i with a_j.

Hence $\quad W_n = M_R^*$

If we define $W_0 = M_R$, then we will have a sequence $W_0, W_1 \ldots W_n$ whose first term is M_R and last term is M_R^*.

Warshall's algorithm gives a procedure to compute each matrix W_k from the previous matrix W_{k-1}. Beginning with the matrix of relation R, we proceed one step at a time, until we reach the matrix of R^*, in n steps. The matrices W_k, being different from powers of the matrix M_R, result in a considerable saving of steps in the computation of the transitive closure of relation R.

Suppose $W_{k-1} = [u_{ij}]$ and $W_k = [v_{ij}]$. If $v_{ij} = 1$, there is a path from a_i to a_j whose interior vertices come from the set $\{a_1, a_2, \ldots a_k\}$. If a_k is not an interior vertex of this path, then all the interior vertices must come actually from $\{a_1, a_2, \ldots a_{k-1}\}$, hence $u_{ij} = 1$.

If a_k is an interior vertex of the path, then we must have the situation.

Since there is a sub path from a_i to a_k whose interior vertices come from $\{a_1, a_2, \cdots a_{k-1}\}$, we must have $u_{ik} = 1$, similarly $u_{kj} = 1$

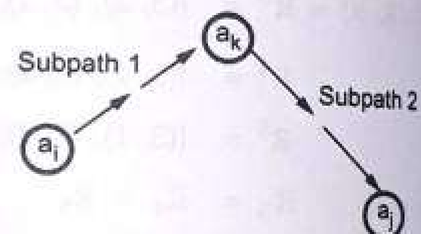


Fig. 6.12

Hence $\quad v_{ij} = 1$ if and only if

1) $\quad u_{ij} = 1$

2) $\quad u_{ik} = 1$ and $u_{kj} = 1$

This is the basis of Warshall's algorithm. If W_{k-1} has 1 in position i, j then by (1) W_k will have 1 in position i, j. A new 1 can be added in position i, j of W_k if and only if column k of W_{k-1} has 1 in position i, and row k of W_{k-1} has 1 in position j.

Thus we have the following procedure for computing W_k from W_{k-1}.

Step 1 : Transfer to W_k, all the 1's in W_{k-1}.

Step 2 : List the locations $p_1, p_2 \ldots$ in column k of W_{k-1} where the entry is 1, and locations $q_1, q_2 \ldots$ in row k of W_{k-1}, where the entry is 1.

Step 3 : Put 1's in all the positions p_i, q_j of W_k (if they are not already there).

Example 6.10 *Use Warshall's algorithm to find transitive closure of R where*

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \{1, 2, 3\}$$

Solution :

$$W_0 = M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 \end{array}$$

and n = 3.

First we find W_1, so that k = 1, W_0 has 1's in location 1 of column 1 and 3 of column 1.

i.e. (1, 1) and (3, 1)

and 1 of row 1 and 3 of row 1

i.e. (1, 1), (1, 3)

$p_1 : \underset{i\ j}{(1, 1)}$ $\quad p_2 = \underset{i\ j}{(3, 1)}$

$q_1 = \underset{i\ j}{(1, 1)}$ $\quad q_2 = \underset{i\ j}{(1, 3)}$

Therefore add (1, 1), (3, 3), (1, 3) in W_1. Thus W_1 is just W_0 with new 1's in the position (1, 1), (3, 3), (1, 3) (if 1's are not already there.)

$$W_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{array}$$

Now to compute W_2, so that k_2, consider 2^{nd} column and 2^{nd} row.

$p_1 : (2, 2)$ $p_2 : (3, 2)$

$q_1 : (2, 2)$

Therefore add (2, 2), (3, 2) in W_1 which are already 1 in W_1

Hence $$W_2 = W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

$p_1 : (1, 3)$ $p_2 : (3, 3)$

$q_1 : (3, 1)$, $q_2 : (3, 2)$, $q_3 : (3, 3)$

Therefore add (1, 1), (1, 2), (1, 3), (3, 1), (3, 2), (3, 3) in W_2 [if 1's are not already there]

Hence $$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \end{array}$$

Hence $R^* = W_3$ is the transitive closure of R.

Therefore

$$R^* = \{(1, 1), (1, 2), (1, 3), (2, 2), (3, 1), (3, 2), (3, 3)\}$$

Example 6.11 *Find the transitive closure of R by Warshall's algorithm.*

Where $A = \{1, 2, 3, 4, 5, 6\}$ and

$R = \{(x - y);\ |x - y| = 2\}$

Solution : $A = \{1, 2, 3, 4, 5, 6\}$

$$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$$

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{n = 6} \end{array}$$

First we find W_1, so that k = 1

$p_1 : (3, 1)$

$q_1 : (1, 3)$

add (p_i, q_i) i.e. (3, 3) in W_0 to get W_1

$$W_1 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Now we compute W_2, so that k = 2

$p_1 : (4, 2)$

$q_1 : (2, 4)$

$\therefore$ add path (4, 4) in W_1 to get W_2

$$W_2 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Now compute W_3, so that k = 3

$p_1 : (1, 3)$, $p_2 : (3, 3)$, $p_3 : (5, 3)$

$q_1 : (3, 1)$, $q_2 : (3, 3)$, $q_3 : (3, 5)$

$\therefore$ add path (p_i, q_i) i.e. (1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5) in W_2 to get W_3.

$$W_3 = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 1 & 0 & 1 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Now for W_4, k = 4

$p_1 : (2, 4), \quad p_2 : (4, 4), \quad p_3 : (6, 4)$

$q_1 : (4, 2), \quad q_2 : (4, 4), \quad q_3 : (4, 6)$

$\therefore$ add path (p_i, q_i) i.e. (2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6,
W_3 to get W_4

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Now for W_5, k = 5

$p_1 : (1, 5), \quad p_2 : (3, 5), \quad p_3 : (5, 5)$

$q_1 : (5, 1), \quad q_2 : (5, 3), \quad q_5 : (5, 5)$

$\therefore$ add path (p_i, q_i), i.e. (1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5
W_4 to get W_5

Here $W_4 = W_5$

Now for W_6, k = 6

$p_1 : (2, 6), \quad p_2 : (4, 6), \quad p_3 : (6, 6)$

$q_1 : (6, 2), \quad q_2 : (6, 4), \quad q_3 : (6, 6)$

$\therefore$ add path (p_i, q_i) i.e. (2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6
W_5 to get W_6 and $W_5 = W_6$. Hence W_6 is R^* (i.e. transitive closure of R)

$$\therefore \qquad R^* = \begin{Bmatrix} (1,1), (1,3), (1,5), (2,2), (2,4), (2,6), \\ (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), \\ (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \end{Bmatrix}$$

Example 6.12 *Find the transitive closure of the relation R on $A = \{1, 2, 3, 4\}$ defined by*

$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)\}$

Solution :

$$M_R = W_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array}_{n=4}$$

First we find W_1, so that k = 1

$p_1 : (2, 1)$

$q_1 : (1, 2), \quad q_2 : (1, 3), \quad q_3 : (1, 4)$

$\therefore$ add (p_i, q_i) i.e. (2, 2), (2, 3), (2, 4) in W_0 to get W_1

$$W_1 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

Now we compute W_2, so that k = 2

$p_1 : (1, 2), \quad p_2 : (2, 2), \quad p_3 : (3, 2), \quad p_4 : (4, 2)$

$q_1 : (2, 1), \quad q_2 : (2, 2), \quad q_3 : (2, 3), \quad q_4 : (2, 4)$

$\therefore$ add paths (p_i, q_i), i.e. (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4) in W_1 to get W_2.

$$W_2 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 1 \\ 4 & 1 & 1 & 1 & 1 \end{array}$$

All positions of W_2 are 1, hence

$W_2 = R^*$

hence $$R^* = \left\{ \begin{array}{l} (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), \\ (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), \\ (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), \\ (4, 4) \end{array} \right\}$$

Example 6.13 *Let* $R = \{(1,4),(2,1),(2,5),(2,4),(4,3),(5,3),(3,2)\}$ *use Warshall's algorithm to find the matrix of transitive closure where* $A = \{1, 2, 3, 4, 5\}$

Solution :

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{n = 5} \end{array}$$

First we find W_1, so that k = 1

W_0 has 1's in location 2 of column 1 i.e. (2, 1) and location 4 of row 1, i.e. (1, 4).

p_1 : (2, 1)

q_1 : (1, 4)

add (p_1, q_1) i.e. (2, 4) in W_1.

Thus W_1 is just W_0 with a new 1 in position (2, 4) (1 is already at (2, 4) position)

$$\text{Hence } W_0 = W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Now we compute k = 2, column 2 has 1 in 3^{rd} location, i.e. (3, 2). row 2 have 1 in 4^{th}, 5^{th} and 6^{th} location.

i.e. (2, 1), (2, 4), (2, 5)

p_1 : (3, 2)

q_1 : (2, 1), (2, 4), (2, 5)

Hence (p_i, q_i) i.e. (3, 1), (3, 4), (3, 5) paths are added to W_1 to get W_2, i.e. W_2 has 1 in location (3, 1), (3, 4) and (3, 5).

$$W_2 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Now we compute k = 3.

p_1 : (4, 3), p_2 : (5, 3)

q_1 : (3, 1), q_2 : (3, 2), q_3 : (3, 4), q_4 : (3, 5)

Hence (p_i, q_i) i.e. (4, 1), (4, 2), (4, 4), (4, 5), (5, 1), (5, 2), (5, 4), (5, 5) paths are added to W_2 to get W_3.

$$W_3 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 0 & 1 & 1 \end{array}$$

Now we compute k = 4.

p_1 : (1, 4), p_2 : (2, 4), p_3 : (3, 4), p_4 : (4, 4), p_5 : (5, 4)

q_1 : (4, 1), q_2 : (4, 2), q_3 (4, 4), q_5 (4, 5)

Hence (p_i, q_i) i.e. (1, 1), (1, 2), (1, 4), (1, 5), (2, 1), (2, 2), (2, 4), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 4), (4, 5), (5, 1), (5, 2), (5, 4), (5, 5) paths are added to W_3 to get W_4.

$$W_4 = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 1 \\ 3 & 1 & 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 & 1 & 1 \\ 5 & 1 & 1 & 0 & 1 & 1 \end{array}$$

Now we compute k = 5.

p_1 : (1, 5), p_2 : (2, 5), p_3 : (3, 5), p_4 : (4, 5), p_5 : (5, 5)

q_1 : (5, 1), q_2 : (5, 2), q_3 : (5, 4), q_4 : (5, 5)

Hence (p_i, q_i), i.e. (1, 1), (1, 2), (1, 4), (1, 5), (2, 1), (2, 2), (2, 4), (2, 5), (3, 1), (3, 2), (3, 4), (3, 5), (4, 1), (4, 2), (4, 4), (4, 5), (5, 1), (5, 2), (5, 4), (5, 5) add to W_4 to get W_5.

$$W_5 = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$W_5 = W_4 = M_{R^*}$$

Hence $$R^* = \begin{bmatrix} (1,1),(1,2),(1,4),(1,5), \\ (2,1),(2,2),(2,4),(2,5), \\ (3,1),(3,2),(3,4),(3,5), \\ (4,1),(4,2),(4,4),(4,5), \\ (5,1),(5,2),(5,4),(5,5) \end{bmatrix}$$

Example 6.14 *Let R be a relation on set A = {1, 2, 3, 4, 5} and*
$R = \{(1,1),(1,2),(1,3),(1,4),(3,1),(3,2),(5,1),(5,2),(5,3),(5,4),(5,5)\}$. *Find transitive closure for R using Warshall's algorithm.*

Solution :

$$M_R = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}_{n = 5} \end{array}$$

First we find W_1, so that k = 1

p_1 : (1, 1), p_2 : (3, 1), p_3 : (5, 1)

q_1 : (1, 1), q_2 : (1, 2), q_3 : (1, 3), q_4 : (1, 4)

$\therefore$ add path (p_i, q_i), i.e. (1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2), (3, 3), (3, 4), (5, 1), (5, 2), (5, 3), (5, 4) in W_0 to get W_1.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Now compute W_2, so that k = 2.

$p_1 : (1, 2)$, $p_2 : (3, 2)$, $p_3 : (5, 2)$

$q_1 :$

Hence no path is added to W_1

Therefore $W_1 = W_2$

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For W_3 , k = 3

$p_1 : (1, 3)$, $p_2 : (3, 3)$, $p_3 : (5, 3)$

$q_1 : (3, 1)$, $q_2 : (3, 2)$, $q_3 : (3, 3)$, $q_4 : (3, 4)$

Therefore add path (p_i, q_i) i.e. (1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2), (3, 3), (3, 4), (5, 1), (5, 2), (5, 3), (5, 4).

Again $W_2 = W_3$

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For W_4 , k = 4

$p_1 : (1, 4)$, $p_2 : (3, 4)$, $p_3 : (5, 4)$

$q_1 :$

Hence no path is added to W_4.

Therefore $W_3 = W_4$

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

For W_5, $k = 5$

$p_1 : (5, 5)$

$q_1 : (5, 1)$, $q_2 : (5, 2)$, $q_3 : (5, 3)$, $q_4 : (5, 4)$, $q_5 : (5, 5)$

Therefore add path (p_i, q_i), i.e. (5, 1), (5, 2), (5, 3), (5, 4), (5, 5) in W_4 to get W_5.

Again $W_4 = W_5$ therefore $W_5 = R^*$

and $R^* = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(3,1),(3,2),(3,3) \\ (3,4),(5,1),(5,2),(5,3),(5,4),(5,5) \end{Bmatrix}$

Example 6.15 *A = {1, 2, 3, 4, 5) and R = {(1,2),(3,4),(4,5),(4,1),(1,1)}. Find its transitive closure by Warshall's algorithm.*

Solution : A = {1, 2, 3, 4, 5}

$$M_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{n = 5} \end{array}$$

First we find W_1, so that k = 1.

W_0 has 1 in location 1 of column 1 i.e. (1, 1) and location 4 of column 1 i.e. (4,1) also 1 in location 1 of row 1, i.e. (1, 1) and location 2 in row 1, i.e. (1, 2).

$p_1 : (1, 1)$, $p_2 : (4, 1)$

$q_1 : (1, 1)$, $q_2 : (1, 2)$

and (p_i, q_i) i.e.(1, 1), (1, 2), (4, 1), (4, 2), in W_0

Hence

$$W_1 = W_0 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now compute W_2, so that k = 2, W_1 has 1 in location 1 and 4 of column 2.

i.e. (1, 2)

p_1 : (1, 2), p_2 : (4, 2)

q_1 :

Hence $W_2 = W_1$

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now computer W_3, so that k = 3

p_1 :

q_1 : (3, 4)

$W_2 = W_3$

$$W_3 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Now computer W_4, so that k = 4.

p_1 : (3, 4)

q_1 : (4, 1), q_2 : (4, 2), q_3 : (4, 5)

and (p_i, q_i) , i.e. (3, 1), (3, 2), (3, 5)

$$W_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

p_i : (3, 5), (4, 5)

q_i :

$$W_4 = W_5$$

Hence W_4 is the transitive closure of the relation matrix M_R.

Hence $R^* = \{(1, 1), (1, 2), (3, 1), (3, 2), (3, 5), (4, 1), (4, 2), (4, 5)\}$

Example 6.16 *Find the transitive closure of R by Warshall's algorithm A = {Set of positive integers ≤ 10} and R = {(a, b) | a divides b}.*

Solution : $$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$R = \left\{\begin{array}{l}(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),\\(1,8),(1,9),(1,10),(2,2),(2,4),(2,6),\\(2,8),(2,10),(3,3),(3,6),(3,9),(4,4),\\(4,8),(5,5),(5,10),(6,6),(7,7),(8,8),\\(9,9),(10,10)\end{array}\right\}$$

$$W_0 = M_R = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10\end{array}\begin{array}{c}\begin{array}{cccccccccc}1&2&3&4&5&6&7&8&9&10\end{array}\\ \begin{bmatrix}1&1&1&1&1&1&1&1&1&1\\0&1&0&1&0&1&0&1&0&1\\0&0&1&0&0&1&0&0&1&0\\0&0&0&1&0&0&0&1&0&0\\0&0&0&0&1&0&0&0&0&1\\0&0&0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1&0&0\\0&0&0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&0&0&1\end{bmatrix}_{n=10}\end{array}$$

The relation R itself is transitive relation on the set of positive integers. Hence

$$R = R^*$$

and M_R is the matrix of transitive closure.

6.7.6 Equivalence Relation

Let R be a relation in a set A. Then R is an equivalence relation on A iff.

i) R is reflexive relation

i.e. $\forall\, a \in A \Rightarrow aRa$

or $(a, a) \in R$

ii) R is symmetric relation

i.e. $(a, b) \in R \Rightarrow (b, a) \in R$

or $a \; R \; b \Rightarrow b \; R \; a$

iii) R is transitive relation

i.e. $a \; R \; b$ and $b \; R \; c \Rightarrow a \; R \; c$

or $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

Digraph of an equivalence relation.

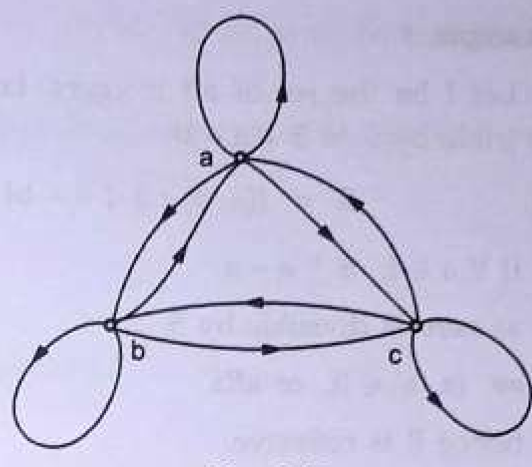


Fig. 6.13

Example 1

Let A be the set of all straight lines in 2-D plane and R be the relation on A defined as aRb iff a is parallel to b.

i) aRa for every $a \in A$ since every straight line is parallel to itself. So R is **reflexive.**

ii) aRb $\Rightarrow$ bRa, since if a line is parallel to line b then b is also parallel to a, hence R is **symmetric.**

iii) aRb, bRc $\Rightarrow$ aRc since if a line is parallel to line b, b line is parallel to c then a is also parallel to c, hence R is **transitive.**

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example 2

Let A be the set of all triangles in 2-D plane. Let R be the relation on A defined as for $a, b \in A$.

aRb iff a is congruent to b. or

$$a \equiv b$$

i) Every triangle is congruent to itself. So aRa, $\forall \; a \in A$ hence R is reflexive.

ii) aRb $\Rightarrow$ bRa, since if triangle a is congruent to b then b is also congruent to a and this shows that R is symmetric.

iii) aRb and bRc $\Rightarrow$ aRc, as if triangle a is congruent to b and triangle b is congruent to c then triangle a is congruent to c. Thus R is transitive.

Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example 3

Let I be the set of all integers. Let R be a relation of I defined as aRb iff a – b is divisible by 5 or 5 | a – b

i.e. $R = \{(a, b) ; 5 \mid a - b\}$

i) $\forall a \in I,\ 5 \mid a - a$

as zero is divisible by 5.

$\Rightarrow (a, a) \in R$ or aRa

hence R is reflexive.

ii) If $a, b \in I$

and $5 \mid a - b$

then 5 also divides b – a.

i.e. $5 \mid b - a$

$\Rightarrow$ bRa or $(b, a) \in R$

hence R is symmetric.

iii) If $a, b, c \in I$

and $5 \mid a - b$ also $5 \mid b - c$.

then $5 \mid a - b + b - c$, i.e. $5 \mid a - c$

hence if aRb, bRc $\Rightarrow$ aRc

hence R is transitive. Being reflexive, symmetric and transitive, R is an equivalence relation.

Note : The same relation R, aRb iff $5 \mid a - b$ is not equivalence relation on the set of natural numbers N.

As $0 \notin N$ so R is not reflexive.

Also if $5 \mid a - b$

i.e. $a - b \in N$

then $b - a \notin N$.

Hence R is not symmetric relation.

But R is transitive

as if $5 \mid a - b$ and $5 \mid b - c$

$\Rightarrow 5 \mid (a - b) + (b - c)$

i.e. $5 \mid a - c$

So R is transitive.

Example 6.17 *If A is the set of all points in a plane, the relation 'at the same distance from the origin as' is an equivalence relation.*

Solution : i) Since the point x is at the same distance from origin as x, i.e. itself.

Therefore xRx, $\forall x \in R$ or $(x, x) \in R$, $\forall x \in R$

Thus, the relation is reflexive.

ii) If xRy or $(x, y) \in R$

$\Rightarrow$ The point x is at the same distance from origin as y.

$\Rightarrow$ y is at the same distance from origin as x.

Thus $(x, y) \in R \Rightarrow (y, x) \in R$

Thus the relation is symmetric.

iii) If x and y points and y and z points are at same distance from origin.

$(x, y) \in R, (y, z) \in R \Rightarrow (x, z) \in R$

i.e. x and z are at same distance from origin. Thus R is transitive relation. Being reflexive, symmetric and transitive relation, R is an equivalence relation.

Example 6.18 *If I be the set of all integers and if the relation R be defined over the set I by aRb iff a – b is an even integers, where a, b ∈ I, show that R is an equivalence relation.*

Solution : i) Since $x - x = 0$ and 0 is an even integer $\Rightarrow$ xRx.

Hence R is reflexive relation.

ii) If $x - y$ is an even integer then $y - x$ is also an even integer

Hence $\Rightarrow$ xRy $\Rightarrow$ yRx

Hence R is symmetric relation.

iii) If xRy, yRz

$\Rightarrow x - y$ in even integer and also $y - z$ is an even integer.

$\Rightarrow (x - y) + (y - z)$ is even integer

$\Rightarrow x - z$ is even integer

$\Rightarrow$ xRz

Hence R is transitive relation. Being reflexive, symmetric and transitive R is an equivalence relation.

Example 6.19 *Let X = {1, 2, ..., 7} and R = {(x, y) | x – y is divisible by 3 }. Show that R is equivalence relation. Draw graph of R.*

Solution :

$$X = \{1, 2, 3, 4, 5, 6, 7\}$$

$$R = \{(x, y) ; x - y = 3k, k \in I\}$$

1. Reflexive relation

$x\,R\,x,\ \forall x \in X$

as $x - x$, i.e. 0 is always divisible by any non-zero finite number and hence divisible by 3.

Hence $x - x = 3 \cdot m, m \in I$

Hence xRx

Therefore R is reflexive relation.

2. Symmetric relation

If xRy $\Rightarrow x - y = 3\,k,\ k \in I.$

$\Rightarrow y - x = 3\,(-k)$

where $-k \in I$

$\Rightarrow$ yRx

Therefore R is symmetric relation.

3. Transitive relation

If xRy and yRz

$\Rightarrow x - y = 3\,k_1$ and $y - z = 3\,k_2$

$\Rightarrow (x - y) + (y - z) = 3\,(k_1 + k_2)$

$\Rightarrow x - z = 3\,(k_1 + k_2)$

$\Rightarrow$ xRz, [as if $k_1 \in I,\ k_2 \in I \Rightarrow k_1 + k_2 \in I$]

Being reflexive, symmetric and transitive relation, R is an equivalence relation.

Graph

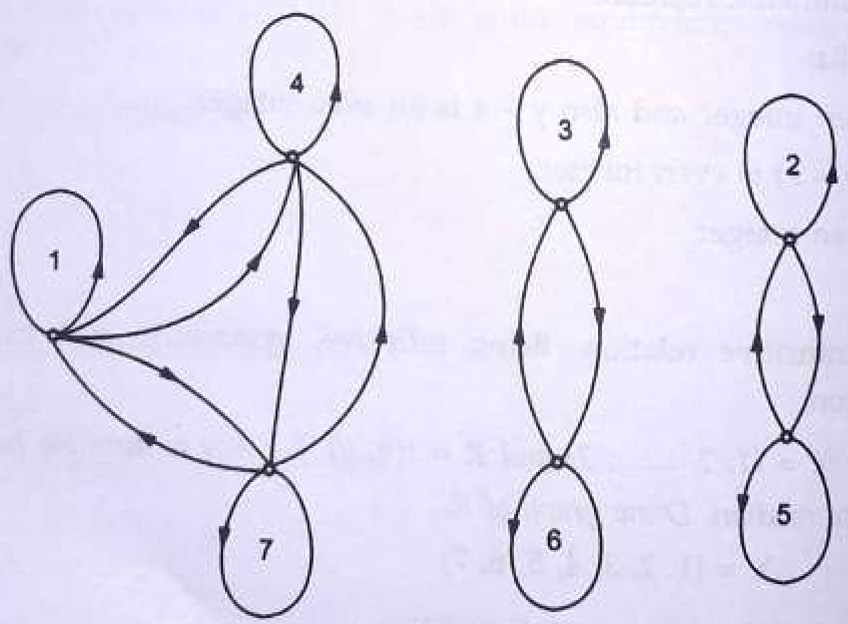


Fig. 6.14

Some more examples of equivalence relations

1. S is the set of all real numbers then a R b iff $a = \pm b$.
2. I is the set of all integers ; $n > 1$ is a fixed integer then aRb iff $a - b$ is a multiple of n.
3. I is the set of all integer then aRb iff $a + b$ is an even integer.
4. S is any non empty set then aRb iff $a = b$.
5. N is the set of natural numbers and $S = \{(a, b) ; a, b \in N\}$ then $(a, b) R (c, d)$ iff $a d = b c$.
6. $S = \{(a, b) ; a, b \in I\}$ and $(a, b) R (c, d)$ iff $a + d = b + c$.

Example 6.20 *Let R be a symmetric and transitive relation on set A. Show that if for every a in A there exist b such that (a, b) is in R, then R is equivalence relation.*

Solution : R is symmetric and transitive relation on set A.

Also $\forall a \in A \Rightarrow (a, b) \in R$

$(a, b) \in R \Rightarrow (b, a) \in R$ [by symmetric property]

also $(a, b) \in R, (b, a) \in R \Rightarrow (a, a) \in R$ [by transitive property]

hence $\forall a \in R, (a, a) \in R$

hence R is a reflexive relation.

Hence R is reflexive, symmetric and transitive relation. Therefore R is equivalence relation.

6.7.6.1 Properties of Equivalence Relation

1. If R and S are equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on set A.

Proof :

To prove that $R \cap S$ is an equivalence relation, we have to prove that,

i) $R \cap S$ is reflexive relation.

ii) $R \cap S$ is symmetric relation.

iii) $R \cap S$ is transitive relation.

i) As R is reflexive.

then $(a, a) \in R, \quad \forall a \in A$.

also S is reflexive.

then $(a, a) \in S, \ \forall a \in A$

$\Rightarrow$ $(a, a) \in R \cap S$, $\forall a \in A$.

Hence $R \cap S$ is reflexive.

ii) R is symmetric.

then $(a, b) \in R \Rightarrow (b, a) \in R$

also S is symmetric

hence $(a, b) \in S \Rightarrow (b, a) \in S$.

Therefore $(a, b) \in R \cap S \Rightarrow (b, a) \in R \cap S$.

Thus $R \cap S$ is symmetric relation.

iii) R and S are transitive relations

hence $(a, b) \in R$, $(b, c) \in R \Rightarrow (a, c) \in R$ and

$(a, b) \in S$, $(b, c) \in S \Rightarrow (a, c) \in S$

Therefore $(a, b) \in R \cap S$, $(b, c) \in R \cap S$

$\Rightarrow (a, c) \in R \cap S$

Hence $R \cap S$ is transitive relation. Being reflexive, symmetric and transitive relation, $R \cap S$ is an equivalence relation.

2. If R and S are equivalence relations then $R \cup S$ need not be an equivalence relation.

Counter Example :

Let $A = \{1, 2, 3\}$

$R = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$

$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

then R is equivalence and also S is an equivalence relation but $R \cup S$ is not equivalence relation,

as $(1, 2) \in R \cup S$, $(2, 3) \in R \cup S$

but $(1, 3) \notin (R \cup S)$

i.e. $R \cup S$ is not transitive relation.

3. If R and S are equivalence relations then $R \cup S$ is an equivalence relation iff $R \subset S$ or $S \subset R$, that is one is contained in other.

6.7.7 Equivalence Classes

Let A be a non-empty set and R be an equivalence relation on set A. Then equivalence class of any element $a \in A$ is denoted by [a] or $\bar{a}$ or A_a and [a] is set of all those elements of A which are related to a.

i.e. $[a] = \{x \,;\, (x, a) \in R \text{ or } xRa\}$

Example 1

Let I be the set of all integers and R is a relation.

$$R = \{(a, b) \,;\, 5 \mid a - b\}, \text{ then}$$

R is an equivalence relation on set A, further if $a \in I$. Then equivalence class of A is set of all those elements of I which are related with a.

i.e. $[a] = \{x \,;\, (x, a) \in R \text{ or } 5 \mid x - a\}$

hence

equivalence class zero

i.e. $[0] = \{x \,;\, 5 \mid x - 0\}$

hence $[0] = \{......, -10, -5, 0, 5, 10, 15, ...\}$

Similarly $[1] = \{x \,;\, 5 \mid x - 1 \text{ or } \frac{x-1}{5} = y \text{ say i.e. } x = 5y + 1\}$

hence $[1] = \{....., -9, -4, 1, 6, 11, 16, ...\}$

$[2] = \{....., -8, -3, 2, 7, 12, ...\}$

$[3] = \{....., -7, -2, 3, 8, 13, ...\}$

$[4] = \{......, -6, -1, 4, 9, 14, ...\}$

$[5] = \{......, -10, -5, 0, 5, 10, 15, ...\}$

From the above example it is clear that $4 \in [4]$, $3 \in [3]$ etc. also two classes are either identical e.g. [0] and [5] or disjoint e.g. [1] and [2].

Hence now let us discuss some properties of equivalence classes.

6.7.7.1 Properties of Equivalence Classes

Let A be a non-empty set and R be an equivalence relation on set A.

$a, b \in A$

Then i) $a \in [a]$

i.e. equivalence classes are non-empty.

ii) if $b \in [a]$ then $[b] = [a]$.

iii) aRb iff $[a] = [b]$, i.e. $(a, b) \in R$ iff $[a] = [b]$.

iv) Either $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

i.e. two equivalence classes are either identical or disjoint.

Proof :

i) As R is equivalence relation on set A

$\Rightarrow$ R is also reflexive

$\Rightarrow aRa \ \forall a \in A$

$\Rightarrow a \in [a]$

as $[a] = \{x \, ; \, xRa, x \in A\}$

ii) If $b \in [a] \Rightarrow bRa$

to show that $[b] = [a]$

We have to show that $[b] \subseteq [a]$ and $[a] \subseteq [b]$

for $[b] \subseteq [a]$

Let $x \in [b] \Rightarrow xRb$

also bRa is given

so $xRb, bRa \Rightarrow xRa$ [by transitive property]

$\Rightarrow x \in [a]$

Hence

$x \in [b] \Rightarrow x \in [a]$

$\Rightarrow [b] \subseteq [a]$ ... (1)

Now Let $y \in [a]$

$\Rightarrow yRa$

also $bRa \Rightarrow aRb$ [as R is symmetric]

also $yRa, aRb \Rightarrow yRb$ [as R is transitive]

So $y \in [a] \Rightarrow y \in [b]$

$\Rightarrow [a] \subseteq [b]$ ... (2)

by equations (1) and (2)

$[a] = [b]$

proof of property (ii)

iii) Suppose $[a] = [b]$

then to prove that aRb

since R is reflexive.

$\Rightarrow$ aRa

$\Rightarrow a \in [a]$

also $[a] = [b]$

$\Rightarrow a \in [b]$

$\Rightarrow$ aRb

Thus $[a] = [b] \Rightarrow aRb$

Conversely to prove

$$aRb \Rightarrow [a] = [b]$$

Let $x \in [a] \Rightarrow xRa$

$xRa, aRb \Rightarrow xRb$ [by transitive property]

$\Rightarrow x \in [b]$

$\Rightarrow [a] \subseteq [b]$ ... (3)

aAlso

if $y \in [b] \Rightarrow yRb$

also $aRb \Rightarrow bRa$ [by symmetric property]

and $yRb, bRa \Rightarrow yRa$

$\Rightarrow y \in [a]$

$\Rightarrow [b] \subseteq [a]$ ... (4)

From equations (3) and (4)

$$[a] = [b]$$

iv) Either $[a] = [b]$ or $[a] \cap [b] = \varnothing$

if no element is common to [a] and [b] then $[a] \cap [b] = \varnothing$ so nothing to prove.

Let one element is common to $[a] \cap [b]$.

i.e. $x \in [a] \cap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow$ xRa and xRb

$\Rightarrow$ aRx and xRb [as R is symmetric]

$\Rightarrow$ aRb [as R is transitive]

$\Rightarrow$ [a] = [b] [by property (iii)]

hence two classes are either identical or disjoint. Hence to show equality of two classes, it is enough to show that one element is common to both the classes.

6.7.8 Partitions

Let S be a non-empty set. A set P = {A, B, C, ...} of non-empty subsets of S will be called a partition of S if

i) $A \cup B \cup C \cup ... = S$, i.e. the set S is the union of the sets of P and

ii) The intersection of every pair of distinct subsets of $S \in P$ is the null set. i.e. if A and $B \in P$, then either $A = B$ or $A \cap B = \emptyset$.

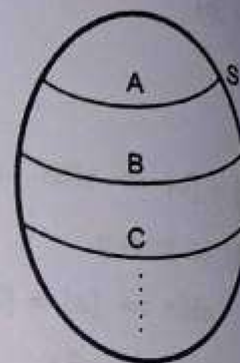


Fig. 6.15

Example 1

Let I be the set of all integers and R = {(a, b) : 5 | a – b} is an equivalence relation on the set I.

Consider the set of five equivalence classes [0], [1], [2], [3], [4] where

$$[0] = \{..., -10, -5, 0, 5, 10, 15, ...\}$$
$$[1] = \{..., -9, -4, 1, 6, 11, 16, ...\}$$
$$[2] = \{..., -8, -3, 2, 7, 12, 17, ...\}$$
$$[3] = \{..., -7, -2, 3, 8, 13, 18, ...\}$$
$$[4] = \{..., -6, -1, 4, 9, 14, 19, ...\}$$

Here i) [0], [1], [2], [3], and [4] are non-empty sets.

ii) The sets [0], [1], [2], [3], [4] are pairwise disjoint.

iii) $I = [0] \cup [1] \cup [2] \cup [3] \cup [4]$

Hence {[0], [1], [2], [3], [4]} is a partition of I.

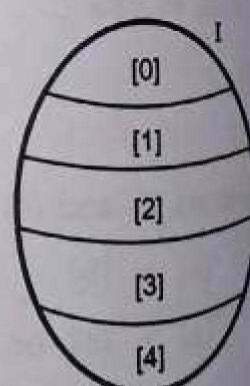


Fig. 6.16

Example 2

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and its subsets $B_1 = \{1, 3\}$, $B_2 = \{7, 8, 10\}$, $B_3 = \{2, 5, 6\}$, $B_4 = \{4, 9\}$.

The set $P = \{B_1, B_2, B_3, B_4\}$ is such that

i) B_1, B_2, B_3, B_4 are all non-empty subsets of S.

ii) $B_1 \cup B_2 \cup B_3 \cup B_4 = S$ and

iii) For any sets B_i, B_j either $B_i = B_j$ or $B_i \cap B_j = \varnothing$.

Hence the set $\{B_1, B_2, B_3, B_4\}$ is a partition of S

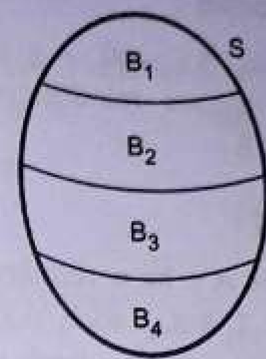


Fig. 6.17

6.7.8.1 Relation Induced by a Partition

Corresponding to any partition of a set S, we can define a relation R on S by the requirement that xRy iff x and y belong to the same subset of S belonging to the partition. The relation R is then said to be induced by the partition.

Example

Consider the subsets

$$A = \{1, 4, 7, \ldots, 25\}$$
$$B = \{2, 5, 8, \ldots, 23\}$$
$$C = \{3, 6, 9, \ldots, 24\}$$

and $$S = \{1, 2, 3, \ldots, 25\}$$

then $A \cup B \cup C = S$, $A \cap B = \varnothing$, $B \cap C = \varnothing$, $A \cap C = \varnothing$, and A, B, C are non-empty sets.

Hence {A, B, C} is partition of S. If R be the relation induced by this partition, then we have xRy iff x and y belong to the same subset A, B, C.

6.7.8.2 Refinement of Partitions

Let A_1 and A_2 be partitions of a non-empty set A. Then A_2 is called refinement of A_1 if every block (element) of A_2 is contained in a block of A_1.

Let $A = \{1, 2, 3, 4\}$

and let $A_1 = \{\{1\}, \{2, 3\}, \{4\}\}$

and $A_2 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

Then A_2 is a refinement of A_1. Let R_1 and R_2 be the equivalence relations induced by A_1 and A_2 respectively.

Then the following theorem relates R_1 and R_1

Theorem :

Let A_1 and A_2 be partition of a non-empty set A and let R_1 and R_2 be the equivalence relations induced by A_1 and A_2 respectively. Then A_2 refines A_1 iff $R_2 \subseteq R_1$.

Proof :

Let A_2 be a refinement of A. We have to prove that $R_2 \subset R_1$. Let aR_2b then $\exists$ some block $T_j' \in A_2$ such that a, b $\in T_j'$. Since A_2 refines A_1. $T_j' \subseteq T_j$ for same block $T_j \in A_1$. Hence a, b $\in T_j$ which implies aR_1b. Hence $R_2 \subseteq R_1$.

Conversely, let $R_2 \subseteq R_1$. We have to prove that A_2 is a refinement of A_1. Let $T_j' \in A_2$ and a $\in T_j'$. Then $T_j' = \{a\}_{R_2}$ let x $\in T_j'$. This implies xR_2a and hence xR_1a since $R_2 \subseteq R_1$. This shows that $\{a\}_{R_2} \subseteq \{a\}_{R_1}$. If $\{a\}_R$ is a block T_j then $T_j' \subseteq T_j$ which indicates that A_2 is the refinement of A_1.

6.7.8.3 Product and Sum of Partitions

Let S be a non-empty set and A_1 and A_2 are partitions of S then $A_1 \cdot A_2$ consists of the set of intersection of every element of A_1 with every element of A_2. Omitting the empty intersections.

Example

$$S = \{1\ 2, 3, 4, 5\}$$
$$A_1 = \{\{1, 2\}, \{3\}, \{4, 5\}\}$$
$$A_2 = \{\{1, 2, 3\}, \{4\}, \{5\}\}$$
then $A_1 \cdot A_2 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}$

Product of partition

$A_1 \cdot A_2$ denote product of partition A_1 and A_2 and let $A_1 \cdot A_2$ be denoted by A' then A' refines both A_1 and A_2.

If T refines both A_1 and A_2, then T refines A'.

Result 1

Let R_1 and R_2 be the equivalence relations induced by partitions A_1 and A_2 respectively. Then the relation $R = R_1 \cap R_2$ induces the product partition $A_1 \cdot A_2$.

Sum of partitions

Let A_1 and A_2 be partitions of a non-empty set S. The sum of A_1 and A_2 denoted by $A_1 + A_2$ is a partition of S such that both A_1 and A_2 refines $A_1 + A_2$. Also if A' is a partition on S such that A_1 and A_2 refines A' then also $A_1 + A_2$ refines A'.

Result 2

Let R_1 and R_2 be equivalence relations on a non-empty set S induced by the partitions A_1 and A_2. Let $R = (R_1 \cup R_2)^*$ the transitive closure of $R_1 \cup R_2$. Then R is an equivalence relation on S and induced by the partition $A_1 + A_2$.

Remark

The product and the sum of two partitions always exists and is unique.

6.8 Fundamental Theorem on Equivalence Relations

An equivalence relation R on a non-empty set S determines a partition of S and conversely, a partition of S defineds an equivalence relation on S.

Proof :

Let R be an equivalence relations on S. Let A be the set of equivalence classes of S with respect to R

i.e. let $A = \{[a] \; ; \; a \in S\}$

where $[a] = \{x : x \in S \text{ and } xRa\}$

Now R is an equivalence relation. Therefore $\forall\, a \in S$, we have a R a. Hence $a \in [a]$ and thus $[a] \neq \varnothing$.

Further every element a of S is an element of equivalence class [a] in A. From this we conclude that $S = \bigcup_{a \in S} [a]$.

Finally, if [a] and [b] are two equivalence classes then either [a] = [b] or $[a] \cap [b] = \varnothing$.

Hence A is a partition of S. Thus an equivalence relation R in S decomposes the set into equivalence classes, any two of which are either identical or mutually disjoint.

Conversely, let $P = [T_a, T_b, T_c, \ldots]$ be any partition of S. If $p, q \in S$, let us define a relation R on S by pRq iff there is a T_i in the partition such that $p, q \in T_i$.

Now $S = T_a \cup T_b \cup T_c \cup \ldots$ therefore $\forall\, x \in S, \exists\, T_i \in P$ such that $x \in T_i$.

Hence $x \in T_i$, $x \in T_i$ means xRx,

thus $\forall\, x \in S$. We have xRx, and thus R is reflexive.

Again if we have xRy, then $\exists\, T_i \in P$ such that $x \in T_i$ and $y \in T_i$.

But $x \in T_i$ and $(y \in T_i \Rightarrow y \in T_i)$ and $x \in T_i$

$\Rightarrow$ yRx.

Therefore R is symmetric.

Finally suppose xRy and yRz. Then by the definition of R, $\exists$ subsets T_j and T_k (not necessarily distinct). Such that $x, y \in T_j$ and $y, z \in T_k$, since $y \in T_j$ and also $y \in T_k$, therefore $T_j \cap T_k \neq \emptyset$. But T_j and T_k belong to a partition of S. Therefore $T_j \cap T_k \neq \emptyset$ implies $T_j = T_k$.

Now $T_j = T_k$ implies $x, z \in T_j$ and consequently we have xRz.

Thus R is transitive. Since R is reflexive, symmetric and transitive, therefore R is an equivalence relation.

6.8.1 Quotient Set

Let S be any non-empty set and let R be an equivalence relation defined on S. The set of mutually disjoint equivalence classes in which S is partitioned relatively to the equivalence relation R, is said to be the Quotient set of S for the equivalence relation and is denoted by S/R or by $\overline{S}$.

Example

The quotient set of I, for the equivalence relation R.

Such that $R = \{(a, b) \; ; \; 5 \mid a - b\}$

then $I/R = \{I_0, I_1, I_2, I_3, I_4\}$

or $I/R = \{[0], [1], [2], [3], [4]\}$

□□□

7 Partial Ordering

Syllabus

Definition, Examples, Simple or Linear Ordering, Totally Ordered Set (Chain), Frequently Used Partially Ordered Relations, Representation of Partially Ordered Sets, Hesse Diagrams, Least & Greatest Members, Minimal & Maximal Members, Least Upper Bound (Supremum), Greatest Lower Bound (infimum), Well-ordered Partially Ordered Sets (Posets). Lattice as Posets, complete, distributive modular and complemented lattices Boolean and pseudo Boolean lattices. (Definitions and simple examples only).

Contents

7.1 Partial Order Relations

A relation R on a set A is called a partial order relation iff R is

i) Reflexive relation

i.e. aRa $\forall$ a$\in$ A

i.e. (a, a) $\in$ R, $\forall$ a$\in$ A

ii) Antisymmetric relation

if aRb and bRa then a = b

i.e. (a, b) $\in$ R, (b, a) $\in$ R $\Rightarrow$ a = b

where a, b $\in$ A

iii) Transitive relation

aRb, bRc $\Rightarrow$ aRc
(a, b) $\in$ R, (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R
where a, b, c $\in$ A

Example 1

1. N is a set of natural numbers. R is a relation defined on set N.

R = {(a, b) ; a divides b or a | b}

a R b iff a | b

a is related with b iff a divides b.

i) $\forall$ a$\in$ N a | a $\Rightarrow$ (a, a) $\in$ R hence R is reflexive relation.

ii) a, b $\in$ N and if a | b and b | a then a=b hence R is antisymmetric relation.

iii) a, b, c $\in$ N if a | b and b | c then a | c i.e. (a, b) $\in$ R, (b, c) $\in$ R $\Rightarrow$ (a, c) $\in$ R hence R is transitive relation. Being reflexive, antisymmetric and transitive, R is a partial order relation.

Example 2

$$S = \{1, 2, 3\}$$

$$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

R is a relation defined on set P (S).

ARB iff A $\subseteq$ B.

Then R is a partial ordering relation on P (S).

i) R is Reflexive

Every set is subset of itself hence A $\subseteq$ A

$\Rightarrow$ ARA, $\forall A \in P(S)$

hence R is reflexive relation.

ii) R is antisymmetric

Let ARB and BRA

$\Rightarrow A \subseteq B$ and $B \subseteq A$

$\Rightarrow \quad A = B$

hence R is antisymmetric relation.

iii) R is transitive

Let ARB and BRC

$\Rightarrow A \subseteq B$ and $B \subseteq C$

$\Rightarrow A \subseteq C$

$\Rightarrow$ ARC

Hence R is transitive. Being reflexive, antisymmetric and transitive relation R is a partial ordering relation.

3. The relation '$\leq$' is a partial order relation on the set of real numbers.
4. The relation '$\geq$' is a partial order relation on the set of real numbers.

7.1.1 Partially Ordered Set or Poset

If A is any non-empty set and R is a partial ordered relation on set A, then the ordered pair (A, R) is called **partially ordered set** or **poset.**

Examples

1. S is a non-empty set and P(S) is power set of S the relation $\subseteq$ is partial order relation on set P(S).

Hence (P(S), $\subseteq$) is known as **poset.**

2. a R b iff a | b (a divides b) is a partial order relation on set of natural numbers set. Hence (N, R) is a **poset.**

7.1.2 Inverse Order and Dual of Poset

If a relation R is a partially ordered relation on a set A then the inverse relation R^{-1} is also partially ordered relation on the set A.

Hence (A, R^{-1}) is a poset, known as **dual of poset** (A, R).

Remark

The relation ≤ is a partial order relation on the set of real numbers. Hence generally '≤' is used to denoted relation on a set A as partial order relation and (A, ≤) is poset.

Example 7.1 *Draw the digraph for the following relation and determine whether the relation is reflexive symmetric, transitive and antisymmetric.*

A = {1, 2, 3, 4, 5, 6, 7, 8} and let xRy whenever y is divisible by x.

Solution : A = {1, 2, 3, 4, 5, 6, 7, 8}

$$R = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7), \\ (1,8),(2,2),(2,4),(2,6),(2,8),(3,3),(3,6), \\ (4,4),(4,8),(5,5),(6,6),(7,7),(8,8) \end{Bmatrix}$$

i) R is reflexive as $\forall$ a $\in$ A, (a, a) $\in$ R

ii) R is not symmetric as (1, 2) $\in$ R but (2, 1) $\notin$ R

iii) R is transitive

iv) R is antisymmetric.

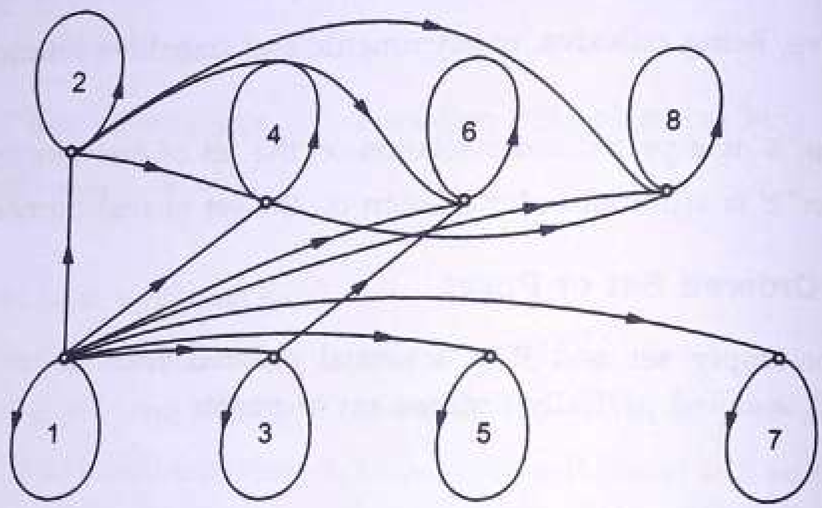


Fig. 7.1

Hence R is partial order relation and A is partial order set.

7.1.3 Hasse Diagrams

Poset can be represented by digraphs. A simpler way of representing poset is Hasse diagram.

Method to find Hasse diagram

From digraph,

1. Omitt loops as relation is reflexive on poset.
2. All arrows that appear on the edges are omitted.

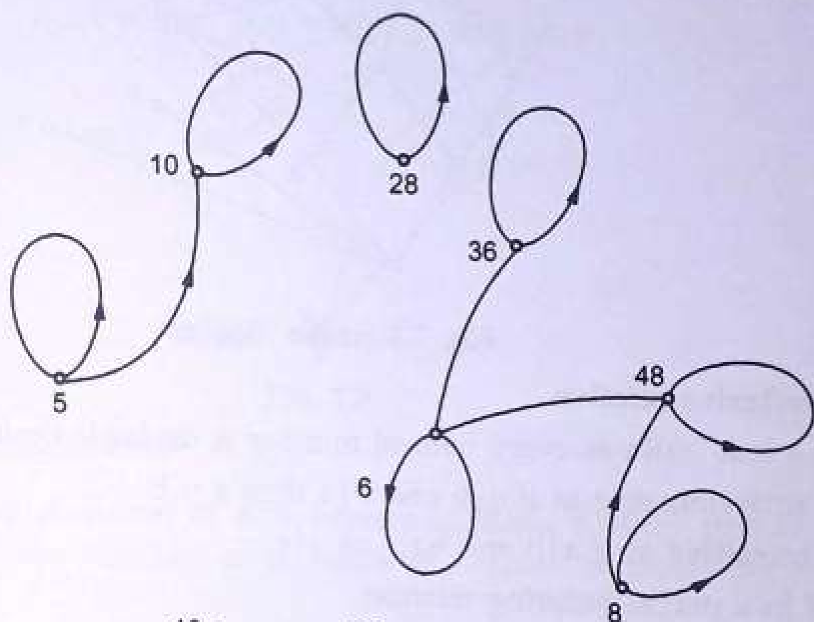
3. Eliminate all edges that are implied by transitive relation.
   e.g. if aRb, bRc, then aRc so eliminate (a, c) edge.
4. An arc pointing upward is drawn from a to b if a ≠ b and aRb.

Example 7.2 *Let R be the relation on the set A.*
A = {5, 6, 8, 10, 28, 36, 48}.
Let R = {(a, b) | a is a divisior of b}. Draw the Hasse diagram.
Compare with digraph. Determine whether R is equivalence relation.

Solution : A = {5, 6, 8, 10, 28, 36, 48}

R = {(5, 5), (6, 6), (8, 8), (10, 10), (28, 28), (36, 36), (48, 48), (5, 10), (6, 36) (6, 48) (8, 48)}

Digraph



Hasse diagram

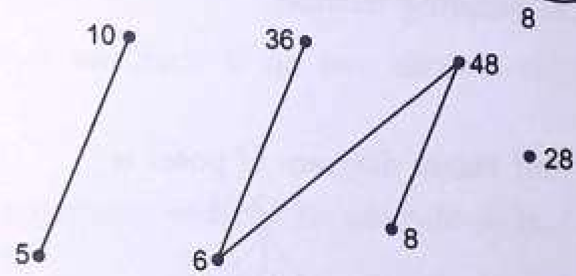


Fig. 7.2

R is reflexive relation, but R is not symmetric relation. Hence R is not equivalence relation.

Example 7.3 *Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24} be orderd by the relation x divides y. Show that the relation is partial ordering and draw the Hasse diagram.*

Solution : $R = \left\{\begin{matrix} (1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6), \\ (1,7), & (1,8), & (1,9), & (1,12), & (1,18), & (1,24), \\ (2,2), & (2,4), & (2,6), & (2,8), & (2,12), & (2,18), \\ (2,24), & (3,3), & (3,6), & (3,9), & (3,12), & (3,18), \\ (3,24), & (4,4), & (4,8), & (4,12), & (4,24), & (5,5), \\ (6,6), & (6,12), & (6,18), & (6,24), & (7,7), & (8,8), \\ (8,24), & (9,9), & (9,18), & (12,12), & (12,24), & (24,24) \end{matrix}\right\}$

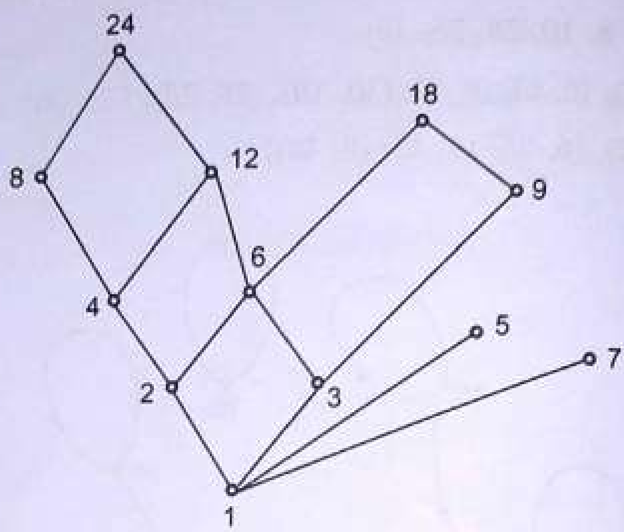


Fig. 7.3 Hasse diagram

1. R is **reflexive relation**

   As $\forall$ a $\in$ A aRa as every natural number is divisible by itself.

2. R in **antisymmetric** as if $a|b$ and $b|a$ then $a = b$.
3. R in **transitive** as if $a|b$ and $b|c \Rightarrow a|c$

Hence R in a partial ordering relation.

Example 1 :

Let S = {1, 2, 3}

(P (S), $\subseteq$) is a **poset** and Hasse diagram of poset is

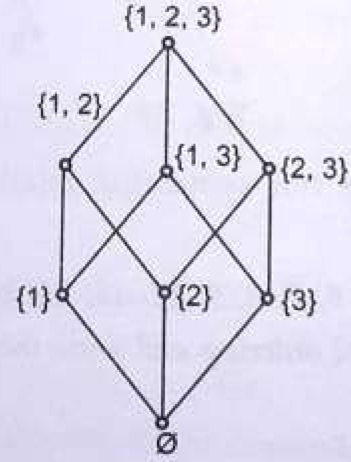


Fig. 7.4

Example 2 :

$$S = \{1, 2, 3, 4\}$$

$$P(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

then (P (S), ⊆) is a **poset** and Hasse diagram of poser is

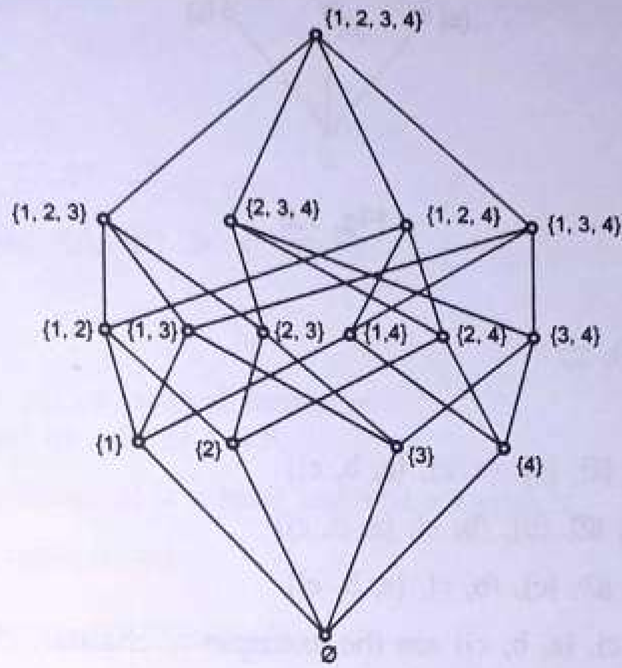


Fig. 7.5

7.2 Chains and Antichains

Let (A, ≤) be a poset. A subset of A is known as **chain** if every pair of elements in the subset are related. The number of elements in a chain is called the length of the chain.

A subset of A is known as antichain if no two distinct elements in a subset are related.

Remark :

Two elements a and b in a partially ordered set are said to be ' not comparable if $a \not\leq b$ (a is not related with b).

7.2.1 Totally Ordered Set

The word partial is used in defining a partial order in a set A because some elements in A need not be comparable. On the otherhand if every two elements in a partially ordered set A are comparable, then the partial order on A is called total order on A and the set A with relation ≤ is known as totally ordered set.

In other words if A itself is a chain the poset (A, ≤) is called a **totally ordered** set or **linearly ordered set.**

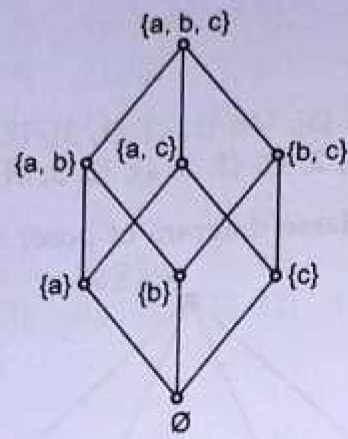


Fig. 7.6

Example 1

S = {a, b, c} Pose (P (S), ⊆)

Hasse diagram

{Ø {a}, {a, b}, {a, b, c}}, {Ø, {a}, {a, c}, {a, b, c}}

{Ø, {b}, {a, b}, {a, b, c}}, {Ø, {b}, {b, c}, {a, b, c}}

{Ø, {c}, {a, c}, {a, b, c}}, {Ø, {c}, {b, c}, {a, b, c}}

{Ø, {a}, {a, b}}, {a, {a, c}, {a, b, c}} are the example of **chains.**

{{a}, {b}}, {{a}, {c}}, {{b}, {c}}, {{a, b}, {a, c}}, {{b, c}, {a, c}}, {{a, b}, {b, c}} are the examples of **antichains.**

Example 2

A = {1, 2, 3, 5, 6, 10, 15, 30}. [Facors of 30] aRb iff a|b (a divides b) then (A, R) is poset.

Hasse diagram

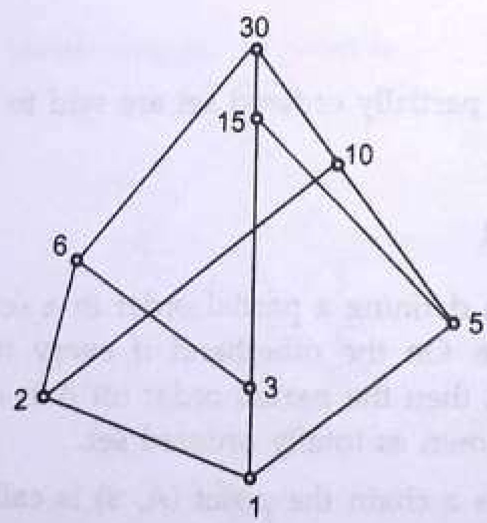


Fig. 7.7

Chains

{1, 2, 6, 30}, {1, 3, 6, 30}, {1, 2, 10, 30}, {1, 3, 15, 30}, {1, 5, 10, 30}, {1, 5, 15, 30}, {2, 6, 30}, {1, 2, 6}, {3, 15, 30}, {1, 3, 15}, {1, 3, 6}, {1, 5, 15}, {1, 5, 10} etc.

Antichains

{2, 3}, {2, 5}, {3, 5}, {6, 10}, {6, 15}, {10, 15}.

Totally ordered relation

Example 1

$$A = \{3, 9, 27, 81, ...\}$$

and aRb iff $a|b$, then (A, R) is a totally ordered relation.

A itself is a chain.

Fig. 7.8

Example 2 : N is the set of natural numbers and R is a relation defined as aRb iff $a \le b$.

then N is a chain and hence N is a total ordered set with $\le$.

i.e. $(N, \le)$ in a totally ordered set.

7.2.2 Maximal and Minimal Elements

Let A be a non-empty set and '$\le$' is partial order relation on A. $(A, \le)$ is poset.

An element $a \in A$ is known as **maximal element** of A if there is no element c in A such that $a \le c$.

An element $b \in A$ is known as **minimal element** of A if there is no element c in A such that $c \le b$.

Example

$$A = \{2, 3, 5, 6, 10, 15, 30\}$$

Hasse diagram of poset

maximal element is 30 and minimal elements are 2, 3, 5.

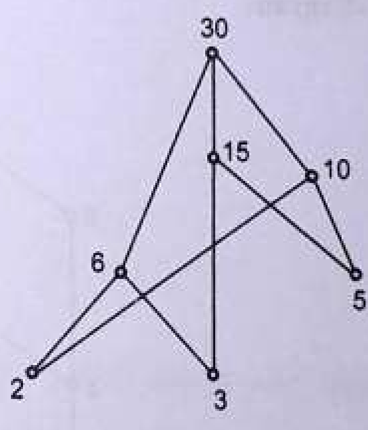


Fig. 7.9

7.2.3 Upper Bounds and Lower Bounds

Let $(A, \le)$ be a poset. For elements $a, b \in A$, an element $c \in A$ is called **upper bound** of a and b if $a \le c$ and $b \le c$.

c is known as **least upper bound** (lub) of a and b if c is an upper bound of a, b, and if there is no other upper bound d of a and b such that d ≤ c. Lub is also known as **supremum.**

Similarly an element e is said to be a **lower bound** of a and b if e ≤ a and e ≤ b; and e is known as **greatest lower bound.** (g*l*b) of a and b if there is no other lower bound f of a and b such that e ≤ f. g*l*b is also known as **infimum.**

Example 1

A = {2, 3, 5, 6, 10, 15, 30, 45}

Hasse diagram

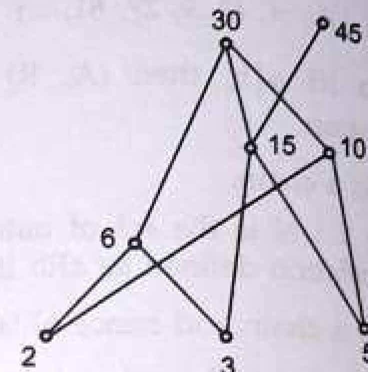


Fig. 7.10

Here 6 and 30 are the upper bounds of 2 and 3. 6 is least upper bound of 2 and 3. Similarly 15, 30 and 45 are the upper bounds of 3 and 5. 15 is the lub. of 3 and 5. Also 10 and 30 are upper bounds of 2 and 5. 10 in the lub of 2 and 5.

Similarly 15, 3, 5 are the lower bounds of 30 and 45 in which 15 is the greatest lower bound.

Example 2

{2, 3, 4, 6, 8, 12, 24, 36}

Hasse diagram

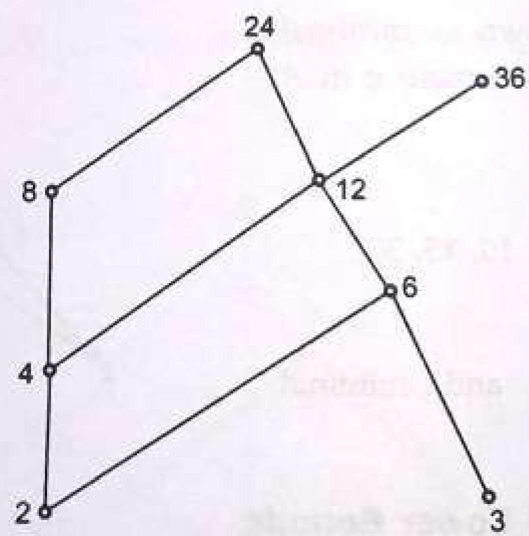


Fig. 7.11

Upper bounds of 2 and 3 are 6, 12 and 24, 36 in which 6 is the least upper bound.

Similarly lower bounds of 24 and 36 are 12, 4, 6, 2, 3 in which 12 is the greatest lower bound.

Example 7.4 *Draw the Hasse diagram of the following sets under the partial ordering relation 'divides' and indicate thoese which are chains :*

1. {2, 4, 12, 24} *2. {1, 3, 5, 15, 30}*

Solution : 1. A = {2, 4, 12, 24}

Hasse diagram

Fig. 7.12

A itself is a chain. Hence (A, ≤) is totally ordered set or linearly ordered set.

2. S = {1, 3, 5, 15, 30}

Hasse diagram

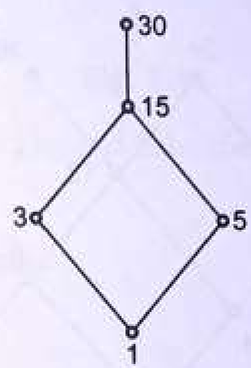


Fig. 7.13

Chains are {1, 3, 15, 30} and {1, 5, 15, 30}

7.3 Lattice

A lattice is a poset (A, ≤) in which every subset {a, b} of A, has a least upper bound and a greatest lower bound.

Example 1

A = {1, 2, 3}

P(A) = {∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

(P(A), ⊆) is a poset.

Hasse diagram

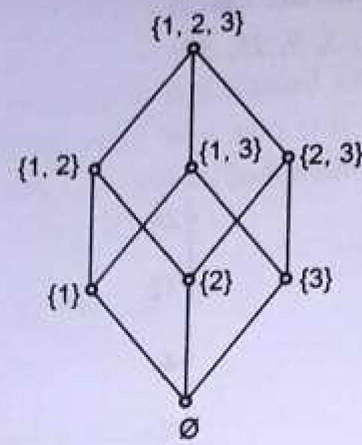


Fig. 7.14

Here every pair of elements has a lub and a glb. Hence (P(A), ⊆) is a lattice.

Example 2

A = {2, 3, 4, 6, 8, 12, 24, 36}

Poset (A, ≤).

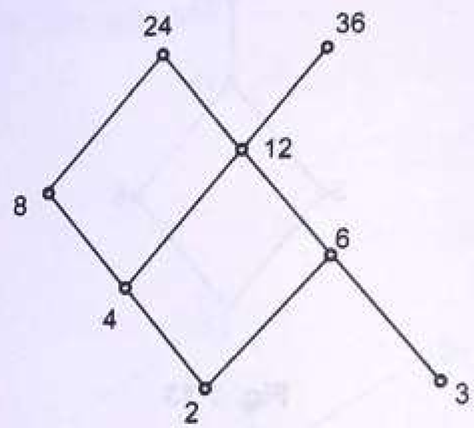


Fig. 7.15

The poset is not a lattice since the pair {2, 3} does not have greatest lower bound and also {24, 36} does not have least upper bound.

7.3.1 Lattice Operators

In lattice **lub** of a and b is denoted by a ∨ b and it is known as a join b. Similarly **glb** of a and b is denoted by a ∧ b and is known as a meet b.

Example 7.5 *Let A be set of factor of positive integer m and relation is divisibility on A.*

i.e. R = {x, y}| x, y ∈ A, x divides y}.

For m = 45 show that Poset (A, ≤) is lattice. Draw Hasse diagram and give join and meet for the lattice.

Solution : A is set of divisors of 45.

Therefore $A = \{1, 3, 5, 9, 15, 45\}$

$R = \{(x, y) ;\ x \mid y,\ x, y \in A\}$

then
$$A = \begin{Bmatrix} (1,1), & (1,3), & (1,5), & (1,9), & (1,15), & (1,45), \\ (3,3), & (3,9), & (3,15), & (3,45), & (5,5), & (5,15), \\ (5,45), & (9,9), & (9,45), & (15,15), & (15,45), & (45,45) \end{Bmatrix}$$

Hasse diagram

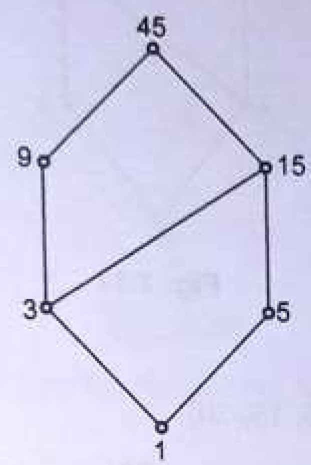


Fig. 7.16

Every pair of elements of A has glb and lub. Hence $(A, \leq)$ is lattice.

Join of a and b is $a \vee b$ (lub).

And meet of a and b is $a \wedge b$ (glb).

Table for join and meet

$\vee$	1	3	5	9	15	45
1	1	3	5	9	15	45
3	3	3	15	9	15	45
5	5	15	5	45	15	45
9	9	9	45	9	45	45
15	15	15	15	45	15	45
45	45	45	45	45	45	45

$\wedge$	1	3	5	9	15	45
1	1	1	1	1	1	1
3	1	3	1	3	3	3
5	1	1	5	1	5	5
9	1	3	1	9	3	9
15	1	3	5	3	15	15
45	1	3	5	9	15	45

Every pair has least upper bound and greatest lower bound, so it is a lattice.

Example 7.6 *Let n be a positive integer S_n, be the set of all divisors of n. Let D denote the relation of 'divison'. Draw the diagrams of lattices for :*

1) $n = 24$, 2) $n = 30$, 3) $n = 6$.

Solution : 1. $N = 24$

$$S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$$

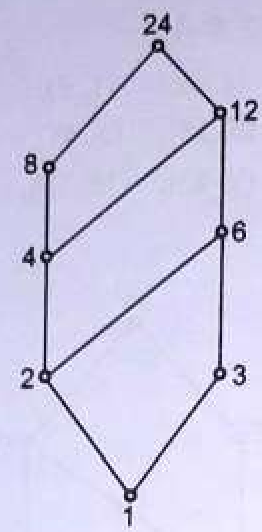


Fig. 7.17

2. $n = 30$

$$S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

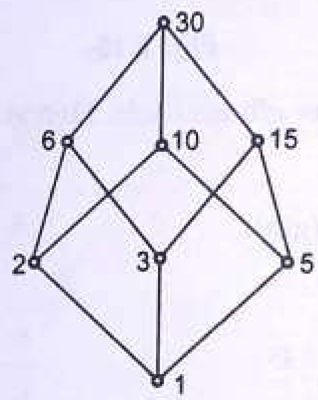


Fig. 7.18

3. $n = 6$

$$S_6 = \{1, 2, 3, 6\}$$

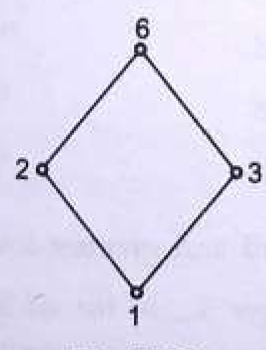


Fig. 7.19

7.3.2 Sublattice

Let (A, ≤) be a lattice. A non-empty subset S of A is called a sublattice of A, if $a \vee b \in S$ and $a \wedge b \in S$ whenever $a \in S$ and $b \in S$.

Example 1 (P(S), ⊆)

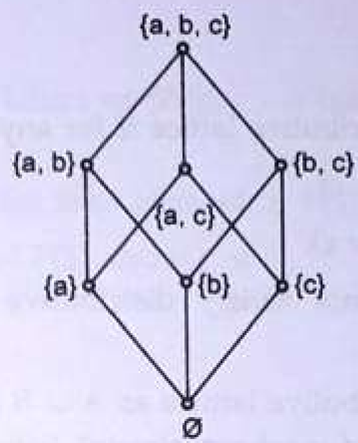


Fig. 7.20

{a, {a, b}, {a, c}, {a, b, c}}, {b, {a, b}, {b, c}, {a, b, c}, {c, {a, c}, {b, c}, {a, b, c}} are sublattice of (P(S), ⊆)

However A_1 = {∅, {a}, {c}, {a, b, c}} is not a sublattice as $a \vee c \notin A_1$

also A_2 = {∅, {a, b}, {a, c}, {a, b, c}} is not a sublattice as

{a, b} ∩ {a, c} = $a \notin A_2$.

7.3.3 Properties of Lattices

Let (A, ≤) be a lattice

then

1) Idempotent properties

i) $a \vee a = a$

ii) $a \wedge a = a$

2) Associative properties

i) $a \vee (b \vee c) = (a \vee b) \vee c$

ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

3) Commutative properties

i) $a \vee b = b \vee a$

ii) $a \wedge b = b \wedge a$

4) Absorption propterties

i) $a \vee (a \wedge b) = a$

ii) $a \wedge (a \vee b) = a$

7.3.4 Types of Lattices

7.3.4.1 Distributive Lattice

A latttice (A, ≤) is called distributive lattice if for any elements a, b, c, ∈ A.

i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Remark : If (A, ≤) does not satisfy distributive property (A, ≤) is known as non-distributive lattice.

Example : (P(S), ⊆) is a distributive latttice as $A \cup B$ (i.e. A union B) and $A \cap B$ (i.e. A intersection B) [or join and meet repectively). Satisfy the distributive property.

7.3.4.2 Bounded Lattice

A lattice (A, ≤) is said to be bounded lattice if greaterst element and least element are in A.

If the greatest element is denoted by 1 and least element is denoted by 0.

then $\forall\ a \in A$

$$0 \leq a \leq 1.$$

also

$$a \vee 0 = a \qquad a \wedge 0 = 0$$

$$a \vee 1 = 1 \qquad a \wedge 1 = a$$

Example :

(P(S), ⊆) is a bounded lattice as $\varnothing \in P(s)$ and $S \in P(S)$. Where S is the greatest element and ∅ is the least element of P(S)

also $\forall\ A \in P(s)$

$\varnothing \subseteq A \subseteq S$

also $A \cup \varnothing = A \qquad A \cap \varnothing = \varnothing$

$A \cup S = S \qquad A \cap S = A$

7.3.4.3 Complemented Lattice

Let (A, ≤) be a bounded lattice. The greatest element of A is say 1 and the least element of A is 0. An element $a' \in A$ is known as complement of $a \in A$.

if i) $a \vee a' = 1$

ii) $a \wedge a' = 0$

A lattice is known as complemented lattice if it is bounded and every element of A has complement.

Example

$$S = \{1, 2, 3\}$$

$(P(s), \subseteq)$ is a complemented lattice as $(P(S), \subseteq)$ is bounded and every element of $P(S)$ has complement.

Remark : 1. Empty set $\varnothing$ is the least element of $P(S)$.

2. S is the greatest element of $P(S)$ also $\{a\}' = \{b, c\}$

$$\{b\}' = \{a, c\}$$

$$\{c\}' = \{a, b\}$$

$$\{a, b\}' = c$$

also $$\{a, b, c\}' = \varnothing$$

and $$\varnothing' = S$$

7.3.4.4 Modular Lattice

A lattice $(A, \leq)$ is known as modular lattice if $\forall$ a, b, c $\in$ A, $a \leq c$

$\Rightarrow$ $a \vee (b \wedge c) = (a \vee b) \wedge c$ [Modular equation]

7.3.4.5 Boolean Lattice

A lattice $(A, \leq)$ is known as Boolean lattice if

1) $(A, \leq)$ is complemented lattice.

2) $(A, \leq)$ is distributive lattice.

8 Recurrence Relations

Syllabus

Introduction, Recursion, Recurrence Relation, Solving, Recurrence Relation.

Contents

8.1 Recurrence Relations

Introduction

Consider the following instruction for generating a sequence

1. Start with 2
2. Give any term, add 5 to get the next term

If we list the terms of the se120

quence, we obtain.

2, 7, 12, 17, 22, ... (1)

The first term is 2 becGTUse of instruction 1. The second term is 7 becGTUse instruction 2 says to add 5 to 2 to get the next term 7. The third term is 12 becGTUse instruction 2 says to add 5 to 7 to get next term 12. By following instructions 1 and 2, we can compute any term in the sequence.

If we denote the sequence (1) as $a_1, a_2, \ldots$, we may rephrase instruction 1 as

$$a_1 = 2 \quad \ldots (2)$$

and we may rephrase instruction 2 as

$$a_n = a_{n-1} + 5 \quad ; \quad n \geq 2 \quad \ldots (3)$$

Taking $n = 2$ in equation (3),

$$a_2 = 2 + 5 = 7$$

taking $n = 3$ in equation (3),

$$a_3 = a_2 + 5 = 7 + 5 = 12$$

Equations (2) and (3) are equivalent to instructions 1 and 2.

Equation (3) is an example of **recurrence relation.**

A recurrence relation defines a sequence by giving the n^{th} value in terms of certain of its predecessors. In order for a recurrence relation such as (3), to define a sequence, a "start-up" value or values, such as (2), must be given. These start up values are known as initial conditions.

Definition

A recurrence relation for the sequence $a_0, a_1, \ldots$ is an equation that relates a_n to certain of its predecessors $a_0, a_1, \ldots a_{n-1}$.

A difference equation is a specific type of recurrence relation.

Example :

Fibonacci numbers

The Fibonacci numbers are defined using the recurrence relation.

$$F_n = F_{n-1} + F_{n-2}$$

with values

$$F_0 = 0$$

$$F_1 = 1$$

Explicitly, recurrence yields the equations

$$F_2 = F_1 + F_0$$

$$F_3 = F_2 + F_1$$

$$F_4 = F_3 + F_2 \text{ etc.}$$

We obtain the sequence of Fibonacci numbers which begins

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

Example

The sequence < 0!, 1!, 2!, ... > satisfies the recurrence relation

$$a_n = n\, a_{n-1}, \quad (n \geq 1)$$

Conversely, given this relation and the initial condition $a_0 = 1$, one can recover the entire sequence by iteration :

$$\begin{aligned} a_n &= n\,[(n-1)a_{n-2}] \\ &= n\,(n-1)\,[(n-2)\,a_{n-3}] \\ &= \ldots\ldots \\ &= n\,(n-1)\,(n-2)\,\ldots.\,3.2.1 \\ &= n! \end{aligned}$$

8.1.1 Homogeneous Linear Recurrence Relations

A k^{th} order linear recurrence relation for the sequence $x_0, x_1, \ldots$ has the form

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_k x_{n-k} + f_n$$

For $n \geq k$, where $a_1, a_2, \ldots a_k$ are constants and $f_k, f_{k+1}, \ldots$ is some given sequence.

We shall restrict our attention to the homogeneous linear recurrence relations : those for which all the f_n are 0. These are the recurrence relations of the form.

$$x_n = a_1x_{n-1} + a_2x_{n-2} + ... + a_kx_{n-k}$$

for $n \geq k$, or equivalently

$x_n - a_1x_{n-1} - a_2x_{n-2} ... - a_kx_{n-k} = 0,$

for $n \geq k$.

Example

Consider the recurrence relation

$$x_n + 2\,x_{n-1} - 15\,x_{n-2} = 0$$

For $n \geq 2$, where $x_0 = 0$ and $x_1 = 1$.

Let $$G(z) = x_0 + x_1z + x_2z^2 + x_3z^3 + ...$$

be the generating function of the sequence $x_0, x_1, x_2, ...$

After multiplying by z and z^2 we find that

$$zG(z) = x_0z + x_1z^2 + x_2z^3 + ...$$

$$z^2G(z) = x_0z^2 + x_1z^3 + ...$$

we now form the expression

$G(z) + 2\,z\,G(z) - 15\,z^2G(z)$. BecGTUse of the recurrence relation most of the terms cancel and so

$$G(z)[1 + 2z - 15\,z^2] = x_0 + (x_1 + 2x_0)z$$

Using the initial values $x_0 = 0$ and $x_1 = 1$ and dividing by $(1 + 2z - 15\,z^2)$ leads to the closed form

$$G(z) = \frac{z}{1 + 2z - 15\,z^2}$$

We have $1 + 2z - 15\,z^2 = (1 - 3z)(1 + 5z)$ and we want to find numbers A and B such that,

$$\frac{z}{1 + 2z - 15\,z^2} = \frac{A}{1 - 3z} + \frac{B}{1 + 5z}$$

Solving this for A and B we find that

$$A = \frac{1}{8} \text{ and } B = -\frac{1}{8}$$

Thus $$G(z) = \frac{1}{8}\left[\frac{1}{1 - 3z}\right] - \frac{1}{8}\left[\frac{1}{1 + 5z}\right]$$

Using the formula for $(1-z)^{-1}$, replacing z by 3z for the first term, and z by $-5z$ for the second term, we obtain,

$$G(z) = \frac{1}{8}[1+3z+3^2z^2+3^3z^3+....]-\frac{1}{8}[1-5z+5^2z^2-5^3z^3+....]$$

The coefficient of z^n in G(z) is x_n and from the expression for G(z) just found we have the solution,

$$x_n = \frac{1}{8}3^n - \frac{1}{8}(-5)^n$$

to our recurrence relation.

This example suggests that the solution to a homogeneous recurrence relation is a combination of n^{th} powers. If we know this to be true we could solve the previous problem as follows.

Step 1 Find the value of λ such that $x_n = \lambda^n$ is a solution to the recurrence relation.

Step 2 Find out what combinations of the solutions found in step 1 satisfy the initial conditions.

Here is how this approach works for the previous example

$$x_n + 2x_{n-1} - 15x_{n-2} = 0$$

For $n \geq 2$ with initial conditions $x_0 = 0$ and $x_1 = 1$. First substitute $x_n = \lambda^n$ in to the recurrence.

After cancellation we see that λ satisfies,

$$\lambda^2 + 2\lambda - 15 = 0$$

That is, $(\lambda-3)(\lambda+5) = 0$ and so the only solutions for λ are 3 and -5.

The next step is to put $x_n = A(3)^n + B(-5)^n$ and find the values of A and B which satisfy the initial conditions.

i.e. we must solve

$$A + B = 0$$

$$3A - 5B = 1$$

therefore $$A = \frac{1}{8} \text{ and } B = -\frac{1}{8}$$

Thus $$x_n = \frac{1}{8}[3^n]-\frac{1}{8}[(-5)^n]$$

as before.

8.1.1.1 First Order Linear Recurrence Relations

The simplest example of a recurrence relation is

$$x_n = a\,x_{n-1}$$

If $\quad G(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + \ldots$

Then $\quad zG(z) = x_0 z + x_1 z^2 + x_2 z^3 + \ldots$

and therefore $(1-az)\,G(z) = x_0$. Thus the generating function for $G(z)$ is $x_0(1-az)^{-1}$

and so $\quad x_n = x_0 a^n$

Indeed we can also prove this by induction.

For n = 1 we have $x_1 = ax_0$ by assumption.

On the other hand, if $n \geq 1$ and $x_n = a^n x_0$

then $\quad x_{n+1} = ax_n = a\,a^n x_0 = a^{n+1} x_0$

Completing the induction step.

8.1.1.2 Second Order Linear Recurrence Relations

$$x_n = a\,x_{n-1} + b\,x_{n-2}, \quad \text{for } n \geq 2$$

If $x'_0, x'_1, x'_2, \ldots$ and $x''_0, x''_1, x''_2 \ldots$ are two sequences which satisfies the recurrence relation and if A and B are constants, then

$$x_n = A\,x'_n + B\,x''_n \quad \text{also satisfies it.}$$

Thus the linear combination $(A\,x'_n + B\,x''_n)$ of the two solutions (x'_n) and (x''_n) is also a solution. This is a very important property and holds in general for all homogeneous linear recurrence relations.

The general solution to second order recurrence relations

We introduce the generating function G(z) of $x_0, x_1, x_2, \ldots$ and observe that

$$G(z) = x_0 + x_1 z + x_2 z^2 + x_3 z^3 + \ldots$$

$$zG(z) = x_0 z + x_1 z^2 + x_2 z^3 + \ldots$$

$$z^2 G(z) = x_0 z^2 + x_1 z^3 + \ldots$$

Therefore

$$G(z) - a\,z\,G(z) - bz^2 G(z) = x_0 + (x_1 - a\,x_0)z$$

From this we obtain the closed form

$$G(z) = \frac{x_0 + (x_1 - a\,x_0)z}{1 - az - bz^2}$$

The next step is to factorize $1 - az - bz^2$ i.e., write

$$(1 - az - bz^2) = (1 - \lambda_1 z)(1 - \lambda_2 z)$$

For some numbers λ_1, and λ_2.

We can now find a partial fraction expansion for G(z). If $\lambda_1 \neq \lambda_2$, this will be of the form

$$G(z) = \frac{A}{1-\lambda_1 z} + \frac{B}{1-\lambda_2 z},$$

For some constants A and B, where as if $\lambda_1 = \lambda_2$, it will be of the form

$$G(z) = \frac{A}{1-\lambda_1 z} + \frac{B}{(1-\lambda_1 z)^2}$$

In the first case, the coefficient of z^n in G(z) is

$$x_n = A\lambda_1^n + B\lambda_2^n$$

and in the second case, the coefficient of Z^n is

$$x_n = A\lambda_1^n + B(n+1)\lambda_1^n = (A+B)\lambda_1^n + n\,B\lambda_1^n$$

To find the values of A and B we need to know some initial conditions such as the values of x_0 and x_1.

The characteristic equation

λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 - a\lambda - b = 0$$

Theorem

If $1 - az - bz^2 = (1-\lambda_1 z)(1-\lambda_2 z)$, then the general solution to the recurrence relation $x_n = a\,x_{n-1} + b\,x_{n-2}$, for $n \geq 2$ is either

$$x_n = A\lambda_1^n + B\lambda_2^n \quad \text{or}$$

$$x_n = C\lambda_1^n + n\,D\,\lambda_1^n \quad \text{according to whether}$$

$$\lambda_1 \neq \lambda_2 \quad \text{or} \quad \lambda_1 = \lambda_2 \quad \text{respectively.}$$

Similarly for k^{th} order linear recurrence relation,

$x_n = A_1 \lambda_1^n + A_2 \lambda_2^n + ... + A_k \lambda_k^n$ is the solution for all distinct roots, $\lambda_1, \lambda_2, ..., \lambda_k$

if all $\quad \lambda_1 = \lambda_2 = ... = \lambda_k = \lambda$

then $\quad x_n = [A_1 n^k + A_2 n^{k-1} + A_3 n^{k-3} + ... + A_k]\lambda^n$ is the solution.

Example 8.1 *Solve the following recurrence relation :*

(i) $a_r - 7a_{r-1} + 10a_{r-2} = 0$ *given that* $a_0 = 0$, $a_1 = 3$

(ii) $a_r - 4a_{r-1} + 4a_{r-2} = 0$ *given that* $a_0 = 1$ *and* $a_1 = 6$

Solution : (i) The characteristics equation is

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 5)(\lambda - 2) = 0$$

$$\lambda = 5, \lambda = 2$$

Therefore the solution of the given recurrence relation is

$$a_r = A(5^r) + B(2^r) \quad ...(1)$$

where A and B are constants

To find A and B putting, r = 0 in equation (1)

$$a_0 = A + B$$

$$\therefore \quad A + B = 0 \quad ...(2)$$

Also putting r = 1 in equation (1)

$$5A + 2B = 3 \quad ...(3)$$

From equation (2) and equation (3)

$$A = 1$$

$$B = -1$$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = 5^r - 2^r$$

(ii) The characteristics equation is

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda = 2, 2$$

Therefore the solution of the given recurrence relation is,

$$a_r = (A+Br)\,2^r$$

Where A and B are constants. ... (4)

To find A and B putting r = 0 in equation (4)

$$a_0 = A$$

$$\Rightarrow \quad A = 1$$

Also putting r = 1 in equation (4)

$$a_1 = (A+B\cdot 1)\,2^1$$

$$6 = (A+B)2$$

$$\Rightarrow \quad A+B = 3$$

or $\quad 1+B = 3$

or $\quad B = 2$

Hence the homogeneous solution of the given recurrence relation is

$$a_r = (1+2r)\,2^r$$

Example 8.2 *Solve the recurrence relation :*

1) $d_n = 2d_{n-1} - d_{n-2}$ *with initial conditions* $d_1 = 1.5, \; d_2 = 3$

2) $d_n = 3d_{n-1} - 2d_{n-2}$ *with initial conditions* $d_1 = -2, d_2 = 4.$

Solution : (1) $d_n - 2d_{n-1} + d_{n-2} = 0, \quad d_1 = \frac{3}{2}, \; d_2 = 3$

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda-1)^2 = 0$$

$$\lambda = 1, 1$$

Therefore the solution of the given recurrence relation is

$$d_n = (A+Bn)\,(1)^n \qquad \dots (1)$$

where A and B are constants

To find A and B, putting n = 1 in equation (1)

$$d_1 = (A+B\cdot 1)$$

$$\frac{3}{2} = A+B \qquad \dots (2)$$

Also putting n = 2 in equation (1)

$$d_2 = A + 2B$$

$$\Rightarrow \quad A + 2B = 3$$

From equation (2) and equation (3),

$$B = \frac{3}{2}, A = 0$$

Hence the homogeneous solution of the given recurrence relation is

$$d_n = \frac{3}{2} n(1)^n$$

2) $d_n + 3d_{n-1} + 2d_{n-2} = 0$, $d_1 = -2$, $d_2 = 4$

The characteristic equation is

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda = -2, -1$$

Therefore the solution of the given recurrence relation is

$$d_n = A(-2)^n + B(-1)^n \quad \text{... (4)}$$

where A and B are constants.

To find A and B putting n = 1 in equation (4)

$$d_1 = A(-2)^1 + B(-1)^1$$

$$-2 = -2A - B$$

$$2A + B = 2 \quad \text{... (5)}$$

Also putting n = 2

$$d_2 = A(-2)^2 + B(-1)^2$$

$$4 = 4A + B \quad \text{... (6)}$$

From equation (5) and equation (6)

$$A = 1, \; B = 0$$

Hence the homogeneous solution of the given recurrence relation is

$$d_n = 1(-2)^n$$

8.1.2 For Non Homogeneous Linear Recurrence Relation [Solution using Generating Function]

Example 8.3 $a_r - 3a_{r-1} = 2,\ r \geq 1$ *with* $a_0 = 1$ *using the generating functions.*

Solution : Given recurrence relation is

$$a_r - 3a_{r-1} = 2$$

Multiplying both sides by z^r, we obtain

$$a_r z^r - 3a_{r-1} z^r = 2z^r$$

Since $r \geq 1$, summing for all r

we get $\sum_{r=1}^{\infty} a_r z^r - 3a_{r-1} \sum_{r=1}^{\infty} z^r$

$$= 2 \sum_{r=1}^{\infty} z^r$$

$$A(z) = a_0 + a_1 z + a_2 z^2 + ...$$

$$\Rightarrow \quad \sum_{r=1}^{\infty} a_r z^r = A(z) - a_0$$

For $\sum_{r=1}^{\infty} a_{r-1} z^r = z \sum_{r=1}^{\infty} a_{r-1} z^{r-1}$

$$= z \sum_{r=0}^{\infty} a_r z^r \qquad \text{[replacing } r-1 \text{ by } r]$$

$$= zA(z)$$

also $\sum_{r=1}^{\infty} z^r = z + z^2 + z^3 + ... + z^r + ... = \frac{z}{1-z}$

[Using sum of infinite terms of geometric progression (G.P)

$$S_n = a + ar + ar^2 + ...$$

then $S_{\infty} = \frac{a}{1-r}, \quad r < 1]$

$$[A(z) - a_0] - 3zA(z) = \frac{2z}{1-z}$$

or $(1 - 3z)\, A(z) = \frac{2z}{1-z} + a_0$

$$a_0 = 1 \text{ (given)}$$

Hence $$A(z) = \frac{2z}{(1-z)(1-3z)} + \frac{1}{1-3z}$$

or $$A(z) = \frac{1+z}{(1-z)(1-3z)}$$

or $$A(z) = \frac{2}{1-3z} - \frac{1}{(1-z)} = 2(1-3z)^{-1} - (1-z)^{-1}$$

$$= 2[1+(3z)+(3z)^2+(3z)^3.....] - [1+z+z^2+z^3+...]$$

$$A(z) = 2\sum_{r=0}^{\infty}(3)^r z^r - \sum_{r=0}^{\infty}(1)^r z^r$$ [Using binomial theorem]

hence $$a_r = 2(3)^r - (1)^r, \; r \geq 0$$

which is the solution of the given recurrence relation.

Remarks : Binomial theorem

$$(x+y)^n = x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + ... + {}^nC_r x^{n-r}y^r + ... + {}^nC_n y^n$$

(when n is a positve integer)

$$(1+a)^n = 1 + na + \frac{n(n-1)}{2!}a^2 + \frac{n(n-1)(n-2)}{3!}a^3 + ...$$

(if n is not necessarily positive integer)

If $n = -1$

$$(1+a)^{-1} = 1 - a + a^2 - a^3 + ...$$

If $a = -b$

$$(1-b)^{-1} = 1 + b + b^2.... = \sum_{r=0}^{\infty} b^r$$

Generating function	Numeric function
$A(z) = \frac{1}{1-az}$	a^r
$A(z) = \frac{1}{(1-z)^2}$	$(r+1)$
$A(z) = \frac{1}{(1-az)^2}$	$(r+1)\,a^r$
$A(z) = \frac{z}{(1-z)^2}$	r

$A(z) = \frac{az}{(1-az)^2}$	$r\,a^r$
[All results are using binomial expensions]	
$A(z) = e^z$	$\frac{1}{n!}$
[Using exponential series]	
$A(z) = (1+z)^n$	$\begin{cases} n c_r, & 0 \le r \le n \\ 0, & r > n \end{cases}$
[Using binomial expansion].	

Total Solutions [For non-homogeneous recurrence relation]

In the previous section, we have seen the method to find the solution of the homogeneous linear recurrence relation. A homogeneous relation is obtained by putting

$$f_n = 0 \text{ in the equation}$$

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \ldots + a_k x_{n-k} + f_n \qquad \ldots (1)$$

If $f_n \neq 0$, the equation (1) is known as non-homogeneous. The solution of equation (1) is called the particular solution.

There is no general procedure for determining the particular solution. It depends on the nature of f(n). The homogeneous solution of the linear recurrence relation is denoted by $a_n^{(h)}$ and the particular solution is denoted by $a_n^{(P)}$. The addition of these two solutions are called a total solution of a given non-homogeneous linear recurrence ralation with the constant coefficients.

Thus, the total solution a_n is given by,

$$a_n = a_n^{(h)} + a_n^{(P)}.$$

The homogeneous solution $a_n^{(h)}$ is found out by finding characteristic roots of the characteristic equation of corresponding homogeneous recurrence relation.

However, there is no general procedure for determining the particular solution of the given recurrence relation.

In some cases, the particular solution can be obtained by the method of inspection of f_n.

Cases

1. If f_n is a constant

$$a_n^{(P)} = P \text{ (constant)}$$

2. If $\quad f_n = C_0 + C_1 n$ (linear function)

then $a_n^{(P)} = P_0 + P_1 n$

3. If $f_n = C_0 + C_1 n + C_2 n^2 + \ldots\ldots + C_k n^k$

(k^{th} degree polynomial)

then $a_n^{(P)} = P_0 + P_1 n + P_2 n^2 + \ldots + \ldots . P_k n^k$

4. $f_n = c \cdot b^n$ provided b is not a characteristic root.

then $a_n^{(P)} = Pb^n$

5. $f_n = c \cdot b^n$ where b is the characteristic root of the equation with multiplicity (m – 1)

then $a_n^{(P)} = P(n)^{m-1} b^n$.

Example 8.4 *Solve the difference equation with given boundary conditions.*

$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r,\ r \geq 2,\ a_0 = 1,\ a_1 = 1.$

Solution : The corresponding homogeneous recurrence relation of the given recurrence relation is given by,

$$a_r - 5a_{r-1} + 6a_{r-2} = 0$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\Rightarrow \quad (\lambda - 3)(\lambda - 2) = 0$$

$$\text{and } \lambda = 3, 2$$

Therefore, the homogeneous solution of the given recurrence relation is

$$a_r^{(h)} = A(3)^r + B(2)^r$$

To find the particular solution we consider the term f(r)

(right hand side of the given equation).

$$f(r) = 2^r + r$$

2 is the root of characteristic equation hence particular solution is,

$$P_1 r^1 (2^r) + (P_2 + P_3 r)$$

Substituting in given equation

$$(P_1 r^1 2^r + P_2 + P_3 r) - 5[P_1(r-1)2^{r-1} + P_2 + P_3(r-1)] + 6[P_1(r-2)\, 2^{r-2} + P_2 + P_3(r-2)] = 2^r + r$$

$$\therefore \quad 2^r\left[P_1 r-\frac{5}{2}P_1(r-1)+\frac{3}{2}P_1(r-2)\right]+2P_2+2P_3 r-7P_3=2^r+r$$

$$\Rightarrow \quad 2^r\left[-\frac{1}{2}P_1\right]+(2P_3)\,r+(2P_2-7P_3)=2^r+r$$

On comparing the coefficients, we get

$$-\frac{P_1}{2} = 1,\ 2P_3=1,\ 2P_2-7P_3=0$$

$$\Rightarrow \quad P_1 = -2,\ P_3=\frac{1}{2},\ P_2=\frac{7}{4}.$$

$$\therefore \quad a_r^{(P)} = -2r(2^r)+\frac{1}{2}+\frac{7}{4}r$$

$\therefore$ Total solution is $a_r = a_r^{(h)} + a_r^{(P)}$

$$a_r = A(3^r)+B(2^r)-2r(2^r)+\frac{1}{2}+\frac{7}{4}r$$

Using boundary conditions $a_0=1,\ a_1=1$,

we get

$$a_0 = A+B+\frac{1}{2}$$

i.e. $$A+B+\frac{1}{2} = 1$$

or $$A+B = \frac{1}{2} \quad \dots (1)$$

For $$r = 1$$

$$a_1 = A(3)^1+B(2)^1-2\cdot 1\cdot 2^1+\frac{1}{2}+\frac{7}{4}\cdot 1$$

$$1 = 3A+2B-4+\frac{1}{2}+\frac{7}{4}$$

$$3A+2B = 5-\frac{9}{4}$$

$$3A+2B = \frac{11}{4} \quad \dots (2)$$

From equation (1) and equation (2),

$$A = \frac{7}{4},\quad B=-\frac{5}{4}$$

$$\therefore \quad a_r = \frac{7}{4}(3^r) - \frac{5}{4}(2)^r - 2r2^r + \frac{1}{2} + \frac{7}{4}r$$

Or

$$a_r = \frac{7}{4}(3^r + r) + 2^r\left(-\frac{5}{4} - 2r\right) + \frac{1}{2}$$

Example 8.5 *Solve the recurrence relation :*

$$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n,\ n \geq 3$$
$$a_0 = 0, a_1 = 0, a_2 = 2$$

Solution : The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

$$\Rightarrow \quad (\lambda + 2)^3 = 0$$

Therefore $\quad \lambda = -2, -2, -2$

Hence the homogeneous solution is

$$a_n^{(h)} = (A_1 n^2 + A_2 n + A_3)(-2)^n$$

The particular solution will be of the form $P(2^n)$. Substituting the given recurrence relation,

we get,

$$P(2^n) + 6\,P(2^{n-1}) + 12\,P(2^{n-2}) + 8\,P(2^{n-3}) = 2^n$$

$$\Rightarrow \quad 2^n P\left[1 + \frac{6}{2} + \frac{12}{4} + \frac{8}{8}\right] = 2^n$$

Or

$$P[1+3+3+1] = 1 \Rightarrow P = \frac{1}{8}$$

$$\therefore \quad a_n^{(P)} = \frac{1}{8}(2^n) = 2^{n-3}$$

Total solution is

$$a_n = a_n^{(h)} + a_n^{(P)} = (A_1 n^2 + A_2 n + A_3)(-2)^n + 2^{n-3}$$

Using the initial conditions

$$a_0 = 0, a_1 = 0, a_2 = 2$$

$$0 = A_3 + \frac{1}{8}$$

$$\Rightarrow \quad A_3 = -\frac{1}{8} \quad \text{... (1)}$$

$$0 = (A_1 + A_2 + A_3)(-2)^1 + 2^{1-3}$$

$$A_1 + A_2 + A_3 = \frac{1}{8} \quad \dots (2)$$

Also

$$2 = (A_1 4 + A_2 \cdot 2 + A_3)(-2)^2 + 2^{2-3}$$

$$2 = (4A_1 + 2A_2 + A_3)(4) + \frac{1}{2}$$

$$\frac{3}{2} = 4(4A_1 + 2A_2 + A_3)$$

$\Rightarrow$ $$4A_1 + 2A_2 + A_3 = \frac{3}{8} \quad \dots (3)$$

From equations (1), (2) and (3) $A_1 = 0$, $A_2 = \frac{1}{4}$, $A_3 = -\frac{1}{8}$

$$a_n = \left(\frac{1}{4}n - \frac{1}{8}\right)(-2)^n + 2^{n-3}$$

Example 8.6 *Solve the recurrence relation*

$a_n = 2a_{n-1} + 3a_{n-2} + 5^n,\ n \geq 2$

Given $a_0 = -2, a_1 = 1.$

Solution : The characteristic equation is

$$\lambda^2 - 2\lambda - 3 = 0$$

$\Rightarrow$ $$(\lambda + 1)(\lambda - 3) = 0$$

Or $$\lambda \quad -1, 3$$

So the homogeneous solution is

$$a_n^{(h)} = A(3)^n + B(-1)^n$$

For particular solution, f_n is of the form 5^n.

Therefore, the particular solution is of the form $P(5^n)$. Substituting in the given recurrence relation,

we get

$$P5^n - 2P5^{n-1} - 3P5^{n-2} = 5^n$$

$\Rightarrow$ $$5^n P\left[1 - \frac{2}{5} - \frac{3}{5^2}\right] = 5^n$$

$\Rightarrow$ $$P\left[\frac{12}{25}\right] = 1$$

Or $$P = \frac{25}{12}$$

Therefore particular solution is

$$a_n^{(P)} = \frac{25}{12}5^n$$

Hence the complete solution is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$\therefore$ $$a_n = A(3^n) + B(-1)^n + \frac{25}{12}(5^n)$$

Using the initial conditions

$$a_0 = -2,\ a_1 = 1, \text{ we get}$$

$$-2 = A + B + \frac{25}{12}$$

Or $$A + B = \frac{-49}{12} \qquad \ldots (1)$$

$$1 = 3A - B + \frac{25}{12}(5)$$

$$3A - B = 1 - \frac{125}{12}$$

$$3A - B = -\frac{113}{12}$$

$\Rightarrow$ $$A = -\frac{131}{24},\quad B = \frac{11}{8}$$

Hence $$a_n = \frac{-131}{24}(3)^n + \frac{11}{8}(-1)^n + \frac{25}{12}(5)^n$$

Example 8.7 *Solve the following recurrence relation :* $a_r - 3a_{r-1} = 2,\ r \geq 1,\ a_0 = 1$

Solution : The characteristic equation is

$$(\lambda - 3) = 0$$

$\Rightarrow$ $$\lambda = 3$$

Hence homogeneous solution is

$$a_r^{(h)} = A(3)^r$$

Particular solution is of the type P (constant)

Hence $$a_r = P$$

Therefore $$a_{r-1} = P$$

Substituting the value of a_r and a_{r-1} in the given recurrence relation.

$$P - 3P = 2$$

$$\Rightarrow \quad -2P = 2$$

$$\Rightarrow \quad P = -1$$

Hence $$a_r = A(3)^r - 1$$

Using initial condition $a_0 = 1$

$$1 = A - 1 \Rightarrow A = 2$$

Hence $$a_r = 2(3^r) - 1$$

□□□

9 Algebraic Structures

Syllabus

Algebraic structures with one binary operation- Semigroup, Monoid, Group, Subgroup, normal subgroup, group Permutations, Coset, homomorphic subgroups, Lagrange's theorem, Congruence relation and quotient structures. Algebraic structures (Definitions and simple examples only) with two binary operation- Ring, Integral domain and field.

Contents

9.1 Definition

An n-ary operation on a non empty set G is a function

$f : (G \times G \times G \text{ ...... n times}) \rightarrow G$

i.e. $f : G^n \rightarrow G$.

If operation is '+' then

$(a_1, a_2, a_3, \text{.......} a_n) \in G \times G \times G$ ..... n times

$\Rightarrow a_1 + a_2 + a_3 + \text{......} a_n \in G$

Similarly for '·'

$a_1 \cdot a_2 \cdot a_3 \text{........} \cdot a_n \in G$.

If n = 1, f is known as unary operation,

If n = 2, f is known as binary operation,

If n = 3, f is known as ternary operation,

and so on.

In the following chapter we are generally dealing with binary operation.

9.1.1 Binary Operation on a Set

Let G be a non empty set.

Then $G \times G : \{(a, b) : a \in G, b \in G\}$.

If $f : G \times G \rightarrow G$ then f is said to be a binary operation on the set G.

'*' is a binary operation on set G.

iff $\forall$ a, b $\in$ G $\Rightarrow$ a * b $\in$ G.

and a * b is unique.

Binary operation on a set G is also known as binary composition.

[Binary means taking 2 elements a and b and combining them with the given operation].

e.g.

3 and 5 $\in$ N

$\Rightarrow$ 3 + 5

= 8 $\in$ N

So '+' is binary operation for the set 'N'.

In other words, the property a * b $\in$ G, $\forall$ a and b $\in$ G is also called closure property.

i.e. G is closed with operation '*'.

e.g.

1. $8 \in N, 7 \in N$

$\Rightarrow 8 + 7$

$= 15 \in N \quad N = \{1, 2, 3, ...\}$

Hence N is closed with respect to addition operation or '+' is called binary operation on the set N.

Similarly

2. $4 \in N, 3 \in N$

$\Rightarrow 4 \cdot 3$

$= 12 \in N$

So N is closed with respect to multiplication operation '·'

3. $7 \in N, 9 \in N$

but $7 - 9$

$= -2 \notin N$

This shows that subtraction operation '–' is not binary operation on the set of N.

4. $4 \in N, 5 \in N$

but $4 \div 5 \notin N$

Hence division is not a binary operation on the set of natural numbers.

Hence addition '+' and multiplication '·' are binary operations for natural numbers set N.

But subtraction '–' and division '÷' are not binary operations on N.

To make subtraction '–' a binary operation we need to extend set N and hence set of integers (I or Z).

$$I = \{......., -3, -2, -1, 0, 1, 2, ...\}$$

As N is subset of I and '+' and '·' are binary operations on N. So are for I.

In case of I,

$4 \in I, 9 \in I$

$\Rightarrow 4 - 9 \in I$

Hence '–' is also a binary, operation on I i.e. '+', '·' and '–' are binary operations on the set of integers I.

But ÷ is not a binary operation on I.

As $5 \in I, 7 \in I$

but $5 \div 7 \notin I$.

So again we need to extend the set of integers and we came to set of rational numbers.

Rational numbers

$$Q = \left\{x\,;\, x = \frac{p}{q}, \text{p and } q \in I, q \neq 0\right\}$$

'+', '–' and '·' are binary operation on Q as they are binary operation for subset I.

For division ÷

$9 \in Q, 8 \in Q$

$\Rightarrow 9 \div 8 \in Q.$

Hence '÷' is also binary operation on the set of rational numbers. But in case of Q, square root of natural number $\sqrt{a}$ where $a \in N$ is not always possible.

So extending the set to real number set $\Re$.

$$\Re = \left\{x\,;\, x = a + b\sqrt{p}\,,\ a \,\&\, b \in Q \text{ and } p \in N \text{ or } p \in Q_{+}\right\}$$

In case of real numbers $\Re$, square root of negative numbers are not defined. So complex numbers set $\mathbb{C}$.

$$\mathbb{C} = \left\{x\,;\, x = a + ib, \text{a and } b \in \Re, i = \sqrt{-1}\right\}$$

Hence $N \subset I \subset Q \subset \Re \subset \mathbb{C}$.

9.1.2 Composition Table or Table for Binary Operation

If S be a finite set, consisting of n elements, then a composition $*$ in S can be described by means of a table consisting of n rows and n columns in which the entry at the intersection of a row headed by an element $a \in S$ and the column headed by an element $b \in S$ is $a * b$. Such tables are called composition tables.

Example

1.

*	a	b	c
a	a * a	a * b	a * c
b	b * a	b * b	b * c
c	c * a	c * b	c * c

2. $S = \{1, \omega, \omega^2\}$, where $\omega^3 = 1$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

3. $S = \{1, -1, i, -i\}$, where $i^4 = (-i)^4 = 1$

$\cdot$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Example 9.1 *Let A = {0, 1}. Is A closed under*

i) Multiplication

ii) Addition

Solution :

+	0	1
0	0	1
1	1	2

$\cdot$	0	1
0	0	0
1	0	1

A is closed with respect to '·' as it is obvious from the table.

A is not closed with respect to '+'

as $1 \in A$, $1 \in A$

but $1 + 1 = 2 \notin A$.

9.1.3 Algebraic Structure

A non empty set G equipped with one or more binary operation is called an algebraic structure.

Suppose ' $*$ ' is a binary operation on G. Then $(G, *)$ is an algebraic structure.

e.g.

$(N, +)$, $(N, \cdot)$, $(N, +, \cdot)$, $(I, +)$, $(I, \cdot)$, $(I, +, \cdot)$, $(I, -)$, $(Q, +)$, $(Q, +)$, $(\Re, \cdot)$, $(\mathbb{C}, +)$ etc are examples of algebraic structures.

But $(N, -)$, $(I, +)$ are not algebraic structures.

9.1.4 Properties of Binary Operations

1. Closure property

$\forall$ a and b $\in$ G, a $*$ b $\in$ G, then G is closed with respect to '$*$' operation

e.g. $5, 9 \in I \Rightarrow 5 + 9 \in I$.

2. Associative property

$\forall$ a, b, c $\in$ G

$$a * (b * c) = (a * b) * c$$

e.g. 2, 3, 5, $\in$ I

then $2 + (3 + 5) = (2 + 3) + 5$

also $2 \cdot (3 \cdot 5) = (2 \cdot 3) \cdot 5$

i.e. '+' and '·' both satisfies associative law on set of integers.

but $(2-3)-5 \neq 2-(3-5)$

Hence '–' is not associative.

Remark : (a $*$ b) $*$ c means first operate a $*$ b and then combine the result with c, but at the same time if it is same as combining a with (b $*$ c). It shows that no need of brackets i.e. we can write elements as a $*$ b $*$ c.

3. Commutative property

$\forall$ a and b $\in$ G

If $a * b = b * a$

then '$*$' on G is said to be commutative.

e.g. $9 + 4 = 4 + 9$

$15 \cdot 3 = 3 \cdot 15$ etc.

Algebraic structures with one algebraic binary operation

9.2 Groupoid

A non empty set G with binary operation '$*$' is called groupoid if the binary operation '$*$' satisfies the following property.

1. Closure property

$\forall$ a, b $\in$ G $\Rightarrow$ a $*$ b $\in$ G

In other words every algebraic structure is groupoid.

e.g. (I, +), (Q, +), ($\mathbb{C}$, –) etc.

9.3 Semi Group

A non empty set G with binary operation ' * ' is known as semi group if the binary operation satisfies the following properties.

1. Closure Property

$\forall a, b \in G \Rightarrow a * b \in G$

2. Associative Property

$\forall a, b, c \in G$

$a * (b * c) = (a * b) * c$

e.g. (N, +), (N, ·), (I, +), (I, ·), (Q, +), (Q, ·) etc. are examples of semi group.

(I, –) is not semi group as '–' is not associative also ÷ on Q is not associative hence (Q, ÷) is not semi group.

9.4 Monoid

A non empty set G with binary operation '*', (G, *) is called monoid if * satisfies.

1. **Closure property** $\forall a, b \in G \Rightarrow a * b \in G$
2. **Associative property** $a * (b * c) = (a * b) * c, \forall a, b, c, \in G$
3. **Existence of identity**

There exists an element $e \in G$ such that $e * a = a = a * e, \forall a \in G$.

The element e is called the identity.

0 is identity element for '+' and $0 \in I$ as $0 + 5 = 5 = 5 + 0$.

So (I, +) is a monoid as I satisfies closure, associative properties for '+' operation and also $0 \in I$.

Other examples of monoid.

(I, ·)

$1 \in I$ and $1 \cdot 6 = 6 = 6 \cdot 1$.

So (I, ·) is a monoid.

$\Rightarrow$ (Q, ·), ($\Re$, ·) and ($\mathbb{C}$, ·) are also monoids.

also (I, +) is a monoid.

$\Rightarrow$ (Q, +), ($\Re$, +) and ($\mathbb{C}$, +) are monoids.

(N, +) is not a monoid,

as $0 \notin N$

(N, ·) is a monoid,

as $I \in N$

In other words semi group with identity element is called **monoid.**

Example 9.2 *Determine whether or not following operations on the set of integers I are associative.*

i) Division ii) Exponentiation.

Solution : i) Division on the set of integers is not associative.

$$(a/b)/c \neq a/(b/c)$$

ii) Exponentiation on the set of integers is not associative.

as $$(a^b)^c \neq a^{(b^c)}$$

Example 9.3 *Show that the algebraic system (A, +) is a monoid, where A is set of integers and + is a binary operation giving addition of two integers.*

Solution : Let A be the set of all integers and '+' is the operation defined on A, then we have to show that (A, +) is a monoid.

1. Closure property

Let $a, b \in A$

then $a + b \in A$ as $a + b$ is also an integer.

2. Associative property

Let $a, b, c \in A$.

then $a + (b+c) = (a + b) + c$

3. Existence of identity

$\forall a \in A \ \exists 0 \in A$

Such that $a + 0 = a = 0 + a$

Therefore (A, +) is a monoid.

Example 9.4 *What are different properties of operations applicable to algebraic systems ?*

Solution : A non-empty set G equipped with one or more binary operations is called an algebraic system.

If G is a non-empty set with binary operation $*$, then (G, $*$) is known as algebraic system or algebraic structure.

Properties of binary operation

1. Commutative property

A binary operation $*$ is known as commutative.

if $\forall$ a, b $\in$ G

$a * b = b * a$

2. Associative property

A binary operation $*$ on G is said to be associative if

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in G.$$

3. Idempotent property

A binary operation $*$ on G is said to satisfy idempotent property if $a * a = a, \forall a \in G$.

9.5 Group

A non empty set equipped with one binary operation $*$ is called group if $*$ satisfies following postulates (or axioms).

1. Closure property

$\forall$ a and b $\in$ G $\Rightarrow$ $a * b \in G$

2. Associative property

$\forall a, b, c \in G \Rightarrow a * (b * c) = (a * b) * c$

3. Existence of identity

There exists an element $e \in G$ such that

$$a * e = a = e * a$$

The element G is called the identity.

4. Existence of inverse

Each element of G possesses inverse, i.e. $\forall$ a $\in$ G $\Rightarrow$ there exists an element $b \in G$ such that

$$a * b = e = b * a$$

Then b is called inverse of a and we write $b = a^{-1}$

So $\quad a * a^{-1} = e = a^{-1} * a$

e.g.

(I, +) is a group.

'+' is closure and associative operation on I.

Also $0 \in I$

Such that $0 + a = a = a + 0$

And $\forall\ a \in I\ \exists -a \in I$

Such that $a + (-a) = 0 = (-a) + a$

Hence (I, +) is a group.

as (I, +) is a group so (Q, +), ($\Re$, +) and ($\mathbb{C}$, +) are also groups.

In other words monoid in which every element has inverse is called group.

Other examples

$(Q_0, \cdot)$

1. Closure property

$\forall\ a, b \in Q_0 \Rightarrow a \cdot b \in Q_0$

2. Associative property

$\forall\ a, b, c \in Q_0 \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$

3. Existence of identity

$1 \in Q_0$

Such that $1 \cdot a = a = a \cdot 1, \quad \forall\ a \in Q_0$

4. Existence of inverse

$\forall\ a \in Q_0\ \exists\ b \in Q_0$

Such that $a \cdot b = 1 = b \cdot a$

e.g. If $\frac{5}{4} \in Q_0$ then $\frac{4}{5}$ is also an element of Q_0.

Such that $\frac{5}{4} \cdot \frac{4}{5} = 1$

also if $5 \in Q_0$ then $\frac{1}{5} \in Q_0$

Such that $5 \cdot \frac{1}{5} = 1 = \frac{1}{5} \cdot 1$

hence $(5)^{-1} = \frac{1}{5}$

There fore $(Q_0, \cdot)$ is a grouop

If $(Q_0, \cdot)$ is a group then $(\Re_0, \cdot)$ and $(\mathbb{C}_0, \cdot)$ are also groups.

Remark :

$Q_0 \rightarrow$ set of rational numbers without zero.

'0' is not having any multiplicative inverse, hence in case of '·' operation, set should be without zero element.

$(I, \cdot)$ is not a group.

as if $2 \in I$, $\frac{1}{2} \notin I$

Such that $2 \cdot \frac{1}{2} = 1$

So $(I, \cdot)$ is a monoid but not a group.

9.6 Abelian Group or Commutative Group

A group with commutative property is known as abelian group or commutative group.

i.e. $\forall$ a, b, $\in$ G, if $a * b = b * a$

' $*$ ' is commutative on G.

Example

1. $(I, +)$ is an abelian group as $(I, +)$ is a group and also $\forall$ a, b $\in$ I.

   $$a + b = b + a$$

2. $(Q, +)$ is an abelian group.

   also $(\Re, +)$ and $(\mathbb{C}, +)$ are examples of abelian group.

3. $(Q_0, \cdot)$, $(\Re_0, \cdot)$ and $(\mathbb{C}_0, \cdot)$ are examples of abelian group.

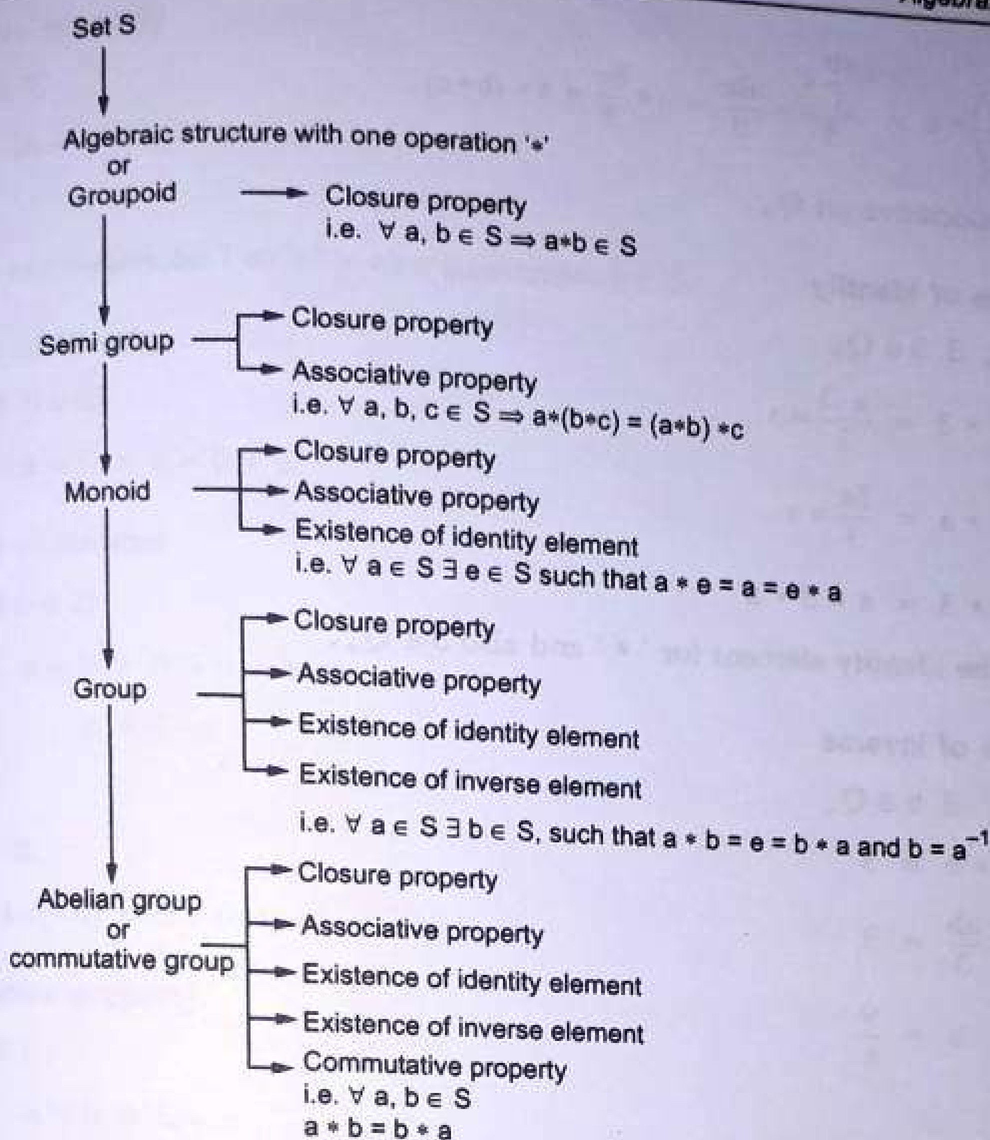


Fig. 9.1

Some More Examples of Groups

Example 9.5 *Let Q_+ be the set of all positive rational numbers and '∗' a binary operation on Q_+ defined by, $a * b = \frac{ab}{3}$ is an abelian group.*

Solution : 1. Closure property

$\forall\, a, b \in Q_+$

$$\Rightarrow \quad a * b = \frac{ab}{3} \in Q_+$$

hence Q_+ is closed with ' ∗ ' operation.

2. Associative property

$\forall\, a, b, c \in Q_+$

$(a * b) * c$

$$= \left(\frac{ab}{3}\right) * c = \frac{\frac{ab}{3}c}{3} = \frac{abc}{9} = a * \frac{bc}{3} = a * (b * c)$$

$\Rightarrow$ ' $*$ ' is associative on Q_+.

3. Existence of identity

$\forall\, a \in Q_+ \;\exists\; 3 \in Q_+$

Such that $a * 3 = \frac{a \cdot 3}{3} = a$

and $3 * a = \frac{3a}{3} = a$

$\Rightarrow \quad a * 3 = a = 3 * a$

Hence 3 is the identity element for ' $*$ ' and also $3 \in Q_+$.

4. Existence of inverse

$\forall\, a \in Q_+ \;\exists\; b \in Q_+$

such that $a * b = 3$

or $\frac{ab}{3} = 3$

$\Rightarrow \quad b = \frac{9}{a}$

and $\frac{9}{a} \in Q_+$

Hence every element possesses inverse.

5. Commutative property

$\forall\, a, b \in Q_+$

$$a * b = \frac{ab}{3} = \frac{ba}{3} = b * a$$

Hence, $(Q_+, *)$ is an abelian group.

Example 9.6 *Set 2I of all even integers with zero is an abelian group with respect to addition.*

Solution : 1. Closure property

$\forall\, a, b \in 2I$

then $a + b \in 2I$

As addition of two even integers is an even integer.

2. Associative property

$\forall$ a, b, c $\in$ 2I

$$(a + b) + c = a + (b + c)$$

As $2I \subset I$

and '+' is associative on I so '+' is also associative for 2I.

3. Existence of identity

$\forall a \in 2I \ \exists \ 0 \in 2I$

Such that $\quad a + 0 = a = 0 + a$

4. Existence of inverse

$\forall a \in 2I \ \exists b \in 2I$

Such that $\quad a + b = 0 = b + a$

then $\quad b = (-a)$

e.g. if $4 \in 2I$

then $\exists -4 \in 2I$.

Such that $\quad 4+(-4) = 0 = (-4)+4$

5. Commutative property

$\forall$ a, b $\in$ 2 I

$$a + b = b + a$$

Hence '+' is commutative on 2I.

So (2I, +) is an abelian group.

RemarkLet S be the set of odd integers.

$$S = \{........, -3, -1, 1, 3, 5, ...\}$$

then S is not closed with '+'

is $\quad 3 + 5 = 8 \notin S$

or addition of two odd integers is an even integer, hence S is not a group.

Example 9.7 $S = \{....,-2m,-m,0,m,2m,3m,....\}$.

Then (S, +) is an abelian group, i.e. the set of multiples of integers by a fixed integer m is an abelian group with respect to addition.

Solution : 1. Closure property

Let a, b be any two elements of S. Then $a = m \cdot r$ and $b = m \cdot s$ where r and s are some integers.

Now $\quad a + b = m r + m s = m (r + s)$

Since r + s is also an integer, therefore m (r + s), i.e. $a + b \in S$. Thus $a + b \in S, \forall a \in S$, $b \in S$. Hence S is closed with respect to addition.

2. Associative property

The elements of S are all integers and addition of integers is associative.

3. Existence of identity

$\exists 0 \in S$, such that

$$0 + a = a = a + 0, \ \forall a \in S$$

Hence 0 is the identity.

4. Existence of inverse

Let mr be an arbitrary element of S where r is some integer.

Then $(-r) m \in S$ $\quad [-r$ is also integer].

Also $(-r) m + r m = (-r + r) m = 0 m = 0$

$\therefore -r m$ is the additive inverse of r m.

Thus every element of S possesses additive inverse.

Hence (S, +) is a group.

Example 9.8 *The set Q_1 of all rational numbers other than 1 with operation $*$ defined by $a * b = a + b - ab$ is an abelian group.*

Solution : 1. Closure property

$a, b \in Q_1$

i.e. $a \neq 1, \ b \neq 1$

$$\Rightarrow \quad a * b = a + b - ab \in Q_1$$

Hence $*$ is closed on Q_1.

2. Associative property

if $a, b, c \in Q_1$

then $(a * b) * c = (a + b - ab) * c = (a + b - ab) + c - (a + b - ab) c$

$$= a + b + c - ab - ac - bc + abc$$

also $a * (b * c)$

$$= a*(b+c-bc) = a+(b+c-bc)-a(b+c-bc)$$
$$= a+b+c-ab-ac-bc+abc$$

Hence $a * (b * c) = (a * b) * c$, hence ' $*$ ' is associative on Q_1.

3. Existence of identity

$\forall a \in Q_1 \; \exists \; e \in Q_1$ i.e. $e \neq 1$.

Such that $a * e = a$

$\Rightarrow \quad a+e-ae = a$

$\Rightarrow \quad e(1-a) = 0$

$\Rightarrow \quad e = 0$ as $a \neq 1$

also $e = 0 \in Q_1$, hence identity is present in Q_1.

4. Existence of inverse

$\forall a \in Q_1 \; \exists \; b \in Q_1$

Such that $a * b = 0$

$a+b-ab = 0$

$a+b(1-a) = 0$

$\Rightarrow \quad b = \dfrac{-a}{1-a}$

also $\quad a \neq 1$

hence $\quad b = \dfrac{-a}{1-a} \in Q_1$

Hence inverse is present for every element of Q_1.

5. Commutative property

$\forall a, b \in Q_1$

$$a * b = a+b-ab = b+a-ba = b * a$$

Hence $(Q_1, *)$ is an abelian group.

Example 9.9 *Set of all $n \times n$ non-singular matrices M having their elements as rational (real or complex) numbers is a non-abelian group with respect to matrix multiplication '.'*

Solution : Let M be the set of all $n \times n$ non-singular matrices with their elements as rational numbers. (A matrix A is said to be non-singular matrix if $|A| \neq 0$).

1. Closure property

Let A, B $\in$ M

i.e. A, B are $n \times n$, non-singular matrices.

hence $|A| \neq 0$ and $|B| \neq 0$

Now $A \cdot B$ is a $n \times n$, matrix.

Also $|A| \neq 0$, $|B| \neq 0 \Rightarrow |AB| \neq 0$.

Hence $A \cdot B$ is non-singular matrix.

Therefore $A \cdot B \in M$

Hence $\forall$ A, B $\in$ M $\Rightarrow$ $A \cdot B \in M$

Thus M is closed with respect to '$\cdot$'

2. Associative property

Matrix multiplication is associative.

3. Existence of identity

Let I_n be a unit matrix of order n, also $|I_n| = 1$, i.e. I_n is a non-singular matrix.

$\Rightarrow$ $I_n \in M$

also $I_n \cdot A = A = A \cdot I_n$

Hence I_n is multiplicative identity element.

4. Existence of inverse

Let $A \in M$

$\Rightarrow$ A is a non-singular matrix.

i.e. $|A| \neq 0$

$\Rightarrow A^{-1}$ exists and $A^{-1} = \dfrac{(\text{adj } A)}{|A|}$

also $A \cdot A^{-1} = I = A^{-1}A$

Hence (M, $\cdot$) is a group.

Note :

1. Matrix multiplication is non-commutative therefore (M, $\cdot$) is non-abelian group.
2. If elements of A are integers or natural numbers, then M is not a group with respect to matrix multiplication. The reason is that all such matrices are not inversible.

For example

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 8 \end{bmatrix}$$

$$|A| = 4 \text{ i.e. } |A| \neq 0$$

and A is 2×2 matrix.

Hence A is non-singular square matrix.

So A^{-1} exists,

and $$A^{-1} = \frac{1}{|A|} (\text{adj } A) = \frac{1}{4}\begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -\frac{3}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$$

Since all the elements of the matrix A^{-1} are not integers, therefore this matrix does not belong to the set M.

Example 9.10 *If $G = \{a + b\sqrt{2}\ ; a, b \in Q\}$, then $(G, +)$ is an abelian group.*

Solution : 1. Closure property

Let x, y be any two elements of G. Then $x = a_1 + b_1\sqrt{2}$, $y = a_2 + b_2\sqrt{2}$ where $a_1, a_2, b_1, b_2 \in Q$.

Now $$x + y = (a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$$

if a_1 and $a_2 \in Q$, then $a_1 + a_2 \in Q$

also $b_1, b_2 \in Q$, then $b_1 + b_2 \in Q$

Hence $x + y \in G$, $\forall\ x, y \in G$ therefore G is closed with respect to addition.

2. Associative property

The elements of G are real numbers and the addition of real numbers is associative.

3. Existence of identity element

$0 + 0\sqrt{2} \in G$ is the identity elememt of G

as $a + b\sqrt{2} + (0 + 0\sqrt{2}) = (a + 0) + (b + 0)\sqrt{2} = a + b\sqrt{2}$

4. Existence of inverse

$a + b\sqrt{2} \in G \Rightarrow -a + (-b)\sqrt{2} \in G$

since $a, b \in Q \Rightarrow -a, -b \in Q$

Now $$(a+b\sqrt{2})+(-a+(-b)\sqrt{2}) = (a+(-a))+(b+(-b))\sqrt{2} = 0+0\sqrt{2}$$
$$= \text{identity}$$

Hence (G, +) is a group.

5. Commutative property

Let $x, y \in G$

$$x = a_1+b_1\sqrt{2}, \quad y = a_2+b_2\sqrt{2}$$

then $$x+y = (a_1+a_2)+(b_1+b_2)\sqrt{2} = (a_2+a_1)+(b_2+b_1)\sqrt{2}$$
$$= (a_2+b_2\sqrt{2})+(a_1+b_1\sqrt{2}) = y+x, \quad \forall x, y \in G$$

Hence (G, +) is an abelian group.

Example 9.11 *Let R = {0°, 60°, 120°, 180°, 240°, 300°} and * = binary operation, so that for a and b in R, a * b is overall angular rotation corresponding to successive rotations by a and then by b. Show that (R, *) is a group.*

Solution :

*	0°	60°	120°	180°	240°	300°
0°	0°	60°	120°	180°	240°	300°
60°	60°	120°	180°	240°	300°	0°
120°	120°	180°	240°	300°	0°	60°
180°	180°	240°	300°	0°	60°	120°
240°	240°	300°	0°	60°	120°	180°
300°	300°	0°	60°	120°	180°	240°

1. Closure property

Obvious from the table,

e.g. $a = 60°$, $b = 180°$

then $a * b = 60° + 180° = 240° \in R$

also $a = 240°$, $b = 180°$

then $a * b = 420° = 360° + 60° = 60° \in R$

as $360° \approx 0°$

2. Associative property

Obvious from the table,

$\forall$ a, b, c $\in$ R

$$a * (b * c) = (a * b) * c$$

e.g.

Let a = 120°, b = 180°, c = 60°

then a * (b * c)

$$= 120° * (180° * 60°) = 120° * (240°)$$

$$= 360° = 0°$$

also (a * b) * c

$$= (120° * 180°) * 60° = (300°) * 60°$$

$$= 360° = 0°$$

Hence a * (b * c) = (a * b) * c

3. Existence of identity

$\forall$ a $\in$ R $\exists$ 0° $\in$ R

such that a * 0° = a°

it is obvious from the table that 0° is the identity element of R.

4. Existence of inverse

$\forall$ a $\in$ R $\exists$ b $\in$ R

Such that a * b = 0°

From the table,

Inverse of 0° = 0°

Inverse of 60° = 300°

Inverse of 120° = 240°

Inverse of 180° = 180°

Inverse of 240° = 120°

Inverse of 300° = 60°

Hence every element possesses inverse.

Therefore (R, *) is a group.

Example 9.12 *G is a group and there exists two relatively prime positive integers m and n such that $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$ for all a, b ∈ G. Prove that G is an abelian group.*

Solution : If m and n are relatively prime ∃ integers p and q such that mp + nq = 1.

To prove that G is an abelian group we have to show that,

$$ab = ba$$

Now $(a^m b^n)^{pm} = (b^n a^m)^{pm}$

and $(a^m b^n)^{nq} = (b^n a^m)^{nq}$

Now $(a^m b^n)^{pm} = (a^m b^n)(a^m b^n) \ldots\ldots (a^m b^n)$ pm times

$= a^m (b^n a^m)(b^n a^m) \ldots\ldots (b^n a^m) b^n$ **(pm − 1)** times

$= a^m (b^n a^m)^{pm-1} b^n = a^m (b^n a^m)^{pm} \cdot (b^n a^m)^{-1} b^n$

$= a^m (b^n a^m)^{pm} \cdot (a^{-m} b^{-n}) b^n$ [as $(a b)^{-1} = b^{-1} a^{-1}$]

$= (b^n a^m)^{pm} \cdot a^m a^{-m}$ [as $a^m b^m = b^m a^n$, ∀ a∈ G]

Similarly $(a^m b^n)^{qn} = (b^n a^m)^{qn}$

$a^m b^n = (a^m b^n)^{mp+nq} = (a^m b^n)^{mp} \cdot (a^m b^n)^{nq}$

$= (b^n a^m)^{mp} \cdot (a^m b^n)^{nq} = (b^n a^m)^{mp} \cdot (b^n a^m)^{nq}$

$= (b^n a^m)^{mp+nq} = b^n a^m \quad \forall\ a, b \in G$

Now consider $ab = a^{pm+qn} b^{pm+qn} = a^{pm} a^{qn} b^{pm} b^{qn}$

$= a^{pm} (a^q)^n (b^p)^m b^{qn} = a^{pm} b^{pm} a^{qn} b^{qn}$

$= b^{pm} a^{pm} b^{qn} a^{qn} = b^{pm} b^{qn} a^{pm} a^{qn}$

$= (b)^{pm+qn} (a)^{pm+qn} = b^1 a^1 = ba$

⇒ $ab = ba$

Hence G is an abelian group.

Example 9.13 *Let G be the set of all non-zero real numbers and let $a * b = \frac{ab}{2}$, show that (G, *) is an abelian group.*

Solution : Let G denote the set of all non-zero real numbers and ' * ' is an operation defined on G.

Such that $a * b = \dfrac{ab}{2}$

We have to show that (G, *) is an abelian group

1. Closure property

a, b$\in$ G then $\frac{ab}{2}$ is also a real number.

Hence $a * b \in G, \forall\ a, b \in G$

Hence G is closed with respect to '* '

2. Associative property

Let a, b, c $\in$ G then

$$(a * b) * c = \frac{ab}{2} * c = \frac{\left(\frac{ab}{2}\right)c}{2}$$

$$= \frac{a}{2}\left(\frac{b\,c}{2}\right) = a * \left(\frac{bc}{2}\right) = a * (b * c)$$

3. Existence of identity

The number e will be the identity element if $e \in G$ and if $e * a = a = a * e, \forall\ a \in G$

$$\text{Now } e * a = a \Rightarrow \frac{ea}{2} = a$$

$$\Rightarrow \left(\frac{a}{2}\right)(e-2) = 0$$

$$\Rightarrow e = 2, \text{ since } a \in G \text{ and } a \neq 0$$

Now 2 $\in$ G, and hence

$$2 * a = \frac{2a}{2} = a = a * 2, \forall\ a \in G$$

Therefore 2 is the identity element.

4. Existence of inverse

Let a be any element of G. If the number b is to be the inverse of a, then we must have,

$b * a = e$

$$\Rightarrow b * a = 2$$

$$\Rightarrow \frac{ba}{2} = 2$$

$$\Rightarrow b = \frac{4}{a}$$

Now a is a real number hence, $b = \frac{4}{a}$ is also a real number and so $\forall\ a \in G\ \exists\ b \in G$. such that,

$$b * a = 2 = a * b$$

Hence inverse of a is $\frac{4}{a}$

5. Commutative property

Let a, b $\in$ G, then

$$a * b = \frac{ab}{2} = \frac{ba}{2} = b * a$$

Hence (G, $*$) is an abelian group.

Example 9.14 *Let G = {even, odd} and binary operation $\oplus$ be defined as,*

$\oplus$	even	odd
even	even	odd
odd	odd	even

Show that (G, $\oplus$) is a group.

Solution : 1. Closure property

Obvious from the table,

e.g. even $\in$ G, odd $\in$ G

$\Rightarrow$ even $\oplus$ odd

= odd $\in$ G

Hence G is closed for $\oplus$

2. Associative property

even $\oplus$ (odd $\oplus$ even)

= even $\oplus$ odd

= odd

also (even $\oplus$ odd) $\oplus$ even

= odd $\oplus$ even

= odd

Hence even $\oplus$ (odd $\oplus$ even) = (even $\oplus$ odd) $\oplus$ even

Hence $\oplus$ is associative operation.

3. Existence of identity

$\exists$ even $\in$ G,

Such that even $\oplus$ even = even

and even $\oplus$ odd = odd = odd $\oplus$ even

Hence even is the identity for $\oplus$

4. Existence of Inverse

$\forall a \in G \; \exists b \in G$

Such that $a \oplus b$ = even

e.g.

$$\text{even} \oplus \text{even} = \text{even}$$
$$\text{odd} \oplus \text{odd} = \text{even}$$

Hence every element is its own inverse

i.e. $(\text{even})^{-1} = \text{even}$

$(\text{odd})^{-1} = \text{odd}$

Hence $(G, \oplus)$ is a group.

Note : If every element is its own inverse then G is an abelian group, hence $(G,\oplus)$ is an abelian group.

9.7 Addition Modulo m

A new type of addition is defined as addition modulo m '$+_m$' and written as $a +_m b$ where a and b are any integers and m is a fixed positive integer.

Hence $a +_m b = r, \quad 0 \le r < m$

where r is the least non-negative remainder when a + b is divided by m.

For example

$$4 +_7 8 = 5$$

Since $4 + 8 = 12 = 1\,(7) + 5$

i.e. 5 is the least non-negative remainder when 4 + 8 is divided by 7.

Similarly $9 +_5 3 = 2$

Thus to find $a +_m b$, we add a and b in the ordinary way and then from the sum we remove integral multiple of m in such a way that the remainder r is either 0 or a positive integer less than m.

9.7.1 Congruent Modulo m

When a and b are two integers such that a – b is divisible by a fixed positive integer then we write,

$$a \equiv b \pmod{m}$$

which is known as "a is congruent to b modulo m".

Hence a ≡ b (mod m) iff a − b is divisible by m.

i.e. a − b = pm where p is some integer.

⇒ a = b + pm.

9.8 Multiplication Modulo m

A new type of multiplication is defined as multiplication modulo m and written as $a \times_m b$ where a and b are integers and m is a fixed positive integer.

Hence $a \times_m b = r, \quad 0 \le r < m$

where r is the least non-negative remainder when a × b is divided by m.

For example

$$4 \times_5 3 = 2$$

As the ordinary product of 4 and 3 is 12 but 12 is greater than 5 and so dividing the product 12 by 5 and the remainder is 2.

also $5 \times_7 6 = 2$ etc.

Result 1

The set {0, 1, 2, ... n − 1} is an additive abelian group of integers modulo n.

Result 2

The set S = {1, 2, ... , p − 1} where p is a prime integer.

S form a multiplicative abelian group of integers modulo p.

Example 9.15 *G = {0, 1, 2, 3, 4, 5, 6, 7} and operation is '$+_8$' addition modulo 8, then $(G, +_8)$ is an abelian group.*

Solution :

$+_8$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

1. Closure property

Obvious from the table, if a and $b \in G \Rightarrow a +_8 b \in G$, if $a + b < 8$ and if $a + b \geq 8$ then, divide the sum by 8 and the remainder will be less than 8 hence is an element of G.

2. Associative property

Obvious from the table.

e.g.,

$$7, 2, 3$$

$$(7 +_8 2) +_8 3$$

$$= 1 +_8 3 = 4$$

also $7 +_8 (2 +_8 3)$

$$= 7 +_8 5 = 4$$

Hence $(7 +_8 2) +_8 3 = 7 +_8 (2 +_8 3)$

3. Existence of identity

$\forall a \in G \; \exists \; 0 \in G$

Such that $a +_8 0 = a = 0 +_8 a$

Hence 0 is the identity element for addition modulo 8 i.e. '$+_8$'.

4. Existence of inverse

$\forall a \in G \; \exists \; b \in G$

Such that $a +_8 b = 0 = b +_8 a$

e.g. Inverse of 0 is 0

Inverse of 1 is 7

Inverse of 2 is 6

Inverse of 3 is 5

Inverse of 4 is 4

Inverse of 5 is 3

Inverse of 6 is 2

Inverse of 7 is 1

Hence every element possesses inverse.

5. Commutative property

$\forall\, a, b \in G$

$$a +_8 b = b +_8 a$$

is obvious from the table.

e.g. 4 and $3 \in G$

$$4 +_8 3 = 7$$

$$3 +_8 4 = 7$$

Hence G is an additive abelian group.

Remark : Set of integers {0, 1, 2, ... n–1} forms an additive abelian group with respect to '$+_n$' addition modulo 'n' where n is any positive integer.

And these are the examples of finite abelian group.

{{0, 1, ... 7}, $+_8$} is an abelian group of order 8 as G contains 8 elements.

Example 9.16 *G = {1, 2, 3, 4, 5, 6}, $\times_7$. '$\times_7$' is multiplication modulo 7.*

Solution :

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

1. Closure property

Obvious from the table.

2. Associative property

Obvious from the table,

e.g. $(3 \times_7 2) \times_7 4$

$$= 6 \times_7 4 = 3$$

Also $3 \times_7 (2 \times_7 4)$

$$= 3 \times_7 1 = 3$$

3. Existence of identity

$\forall a \in G \; \exists 1 \in G$

such that $a \times_7 1 = a = 1 \times_7 a$

4. Existence of inverse

$\forall a \in G \; \exists b \in G$

Such that $a \times_7 b = 1 = b \times_7 a$

e.g. Inverse of 1 is 1

Inverse of 2 is 4

Inverse of 3 is 5

Inverse of 4 is 2

Inverse of 5 is 3

Inverse of 6 is 6.

Hence every element possesses inverse.

5. Commutative property

Obvious from the table,

e.g. $4 \times_7 3 = 5$

also $3 \times_7 4 = 5$

Hence G = {1, 2, 3, 4, 5, 6} form an abelian group with respect to multiplication modulo 7. i.e. '$\times_7$'.

Remark : G = {1, 2, 3, 4, 5, 6, 7}, $\times_8$ is not a group.

as $2 \times_8 4 = 0$

but $0 \notin G$

Hence closure property is not satisfied.

If 0 is included in the set G then 0 cannot have inverse and hence G can not form a group.

i.e. multiplication modulo a composite integer cannot form a group. Only multiplication modulo a prime integer can form a group.

Example 9.17 *Show that $< Z_6, + >$ is abelian group.*

Solution : Let Z_6 be the set of residue classes modulo 6.

i.e. $Z_6 = \{[0], [1], [2], [3], [4], [5]\}, +$

+	{0}	{1}	{2}	{3}	{4}	{5}
{0}	{0}	{1}	{2}	{3}	{4}	{5}
{1}	{1}	{2}	{3}	{4}	{5}	{0}
{2}	{2}	{3}	{4}	{5}	{0}	{1}
{3}	{3}	{4}	{5}	{0}	{1}	{2}
{4}	{4}	{5}	{0}	{1}	{2}	{3}
{5}	{5}	{0}	{1}	{2}	{3}	{4}

1. Closure property

Obvious from the table,

e.g. $\{2\}, \{3\} \in Z_6$

$\Rightarrow \quad \{2\} + \{3\} = \{5\} \in Z_6$

2. Associative property

Obvious from the table,

e.g. $\{1\}, \{4\}, \{5\} \in Z_6$

$$(\{1\}+\{4\})+\{5\}) = \{5\}+\{5\}$$

$$= \{4\}$$

also $\{1\} + (\{4\} + \{5\})$

$$= \{1\} + \{3\} = \{4\}$$

Hence $\quad (\{1\} + \{4\}) + \{5\} = \{1\} + (\{4\} + \{5\})$

3. Existence of identity

$\{0\}$ is the identity element of Z_6.

4. Existence of inverse

$\forall \{a\} \in Z_6 \; \exists \{b\} \in Z_6$

Such that $\{a\} + \{b\} = \{0\}$

Then $\{b\} = \{a\}^{-1}$

Inverse of $\{1\}$ is $\{5\}$

Inverse of $\{2\}$ is $\{4\}$

Inverse of $\{3\}$ is $\{3\}$

Inverse of $\{4\}$ is $\{2\}$

Inverse of {5} is {1}.

Thus every element is inversible.

Hence $(Z_6, +)$ is a group.

5. Commutative property

Obvious from the table,

e.g. $\{4\} + \{3\} = \{1\} = \{3\} + \{4\}$

Hence $(Z_6, +)$ is an abelian group.

9.9 Properties of Groups

1. Identity element is unique

Proof : Let $(G, *)$ be a group and e and e' be two identities of $(G, *)$.

then we have

$e * e' = e$ as e' is the identity.

Also $e * e' = e'$ as e is also identity.

But $e * e'$ is unique element of $(G, *)$

hence $e * e' = e$ and $e * e' = e' \Rightarrow e = e'$.

Hence the identity element is unique.

2. Inverse element is unique

Proof : Let a be any element of $(G, *)$ and let e be the identity element.

Suppose b and c are two inverse of a

then $a * b = e = b * a$

and $a * c = e = c * a$

$\Rightarrow b * (a * c) = b * e = b$

also $(b * a) * c = e * c = c$

Using associative property

$$b = c$$

Therefore inverse of an element is unique.

3. Cancellation laws hold good in a group

If a, b, c are any elements of $(G, *)$, then

$a * b = a * c \Rightarrow b = c$ [Left cancellation law]

and $b * a = c * a \Rightarrow b = c$ [Right cancellation law]

Proof : If $a \in G$ then $\exists\ a^{-1} \in G$, such that

$$a * a^{-1} = e = a^{-1} * a$$

Now $$a * b = a * c$$

$\Rightarrow$ $$a^{-1} * (a * b) = a^{-1} * (a * c)$$

$\Rightarrow$ $$(a^{-1} * a) * b = (a^{-1} * a) * c$$ [Associative property]

$\Rightarrow$ $$e * b = e * c$$

$\Rightarrow$ $$b = c$$

Also $$b * a = c * a$$

$\Rightarrow$ $$(b * a) * a^{-1} = (c * a) * a^{-1}$$

$\Rightarrow$ $$b * (a * a^{-1}) = c * (a * a^{-1})$$ [Associative property]

$\Rightarrow$ $$b * e = c * e$$ $[a * a^{-1} = e]$

$\Rightarrow$ $$b = c$$ [e is identity]

4. If the inverse of a is a^{-1}, then the inverse of a^{-1} is a

i.e. $$(a^{-1})^{-1} = a$$

Proof : Let e be the identity element of (G, ∗), then we have

$$a^{-1} * a = e = a * a^{-1}$$

$$a^{-1} * a = e$$

$\Rightarrow$ $$(a^{-1})^{-1} * [a^{-1} * a] = (a^{-1})^{-1} * e$$

$\Rightarrow$ $$[(a^{-1})^{-1} * a^{-1}] * a = (a^{-1})^{-1} * e$$ [By associative property]

$\Rightarrow$ $$[e] * a = (a^{-1})^{-1}$$ [As $(a^{-1})^{-1}$ is inverse of a^{-1}]

$\Rightarrow$ $$a = (a^{-1})^{-1}$$

Remark : If (G, +) is a group, then

$$-(-a) = a$$

5. $(a * b)^{-1} = b^{-1} * a^{-1}$, $\forall$ a, b $\in$ (G, ∗)

Proof : Let a and b be two elements of (G, ∗) then a^{-1} and b^{-1} are inverse of a and b respectively, then

$$a^{-1} * a = e = a * a^{-1}$$

$$b^{-1} * b = e = b * b^{-1}$$

also

Now $(a * b) * (b^{-1} * a^{-1}) = ((a * b) * b^{-1}) * a^{-1}$ [By associative property]

$$= (a * (b * b^{-1})) * a^{-1} = (a * e) * a^{-1}$$

$$= a * a^{-1} = e$$

Hence $b^{-1} * a^{-1}$ is inverse of $(a * b)$

i.e. $$(a * b)^{-1} = b^{-1} * a^{-1}$$

Remark : 1. If $(G, *)$ is an abelian group, then

$$a * b = b * a$$

Hence $$(b * a)^{-1} = b^{-1} * a^{-1}$$

Therefore $$(a * b)^{-1} = a^{-1} * b^{-1}$$

2. If a, b, c, d, ... h, are elements of G

then $(a * b * c * d \ldots\ldots * h)^{-1} = h^{-1} * \ldots\ldots * d^{-1} * c^{-1} * b^{-1} * a^{-1}$

3. If $(G, +)$ is a group

then $$-(a + b) = (-b) + (-a)$$

9.10 Some Results on Abelian Group

1. If every element of a group G is its own inverse, then G is abelian group.

Solution : Let $a, b \in G$, then $a = a^{-1}$ and $b = b^{-1}$

Also $a, b \in G \Rightarrow a\,b \in G$

Hence $$(a\,b)^{-1} = ab$$

But $$(ab)^{-1} = b^{-1}\,a^{-1}$$

Hence $$b^{-1}\,a^{-1} = ab$$

and $$b^{-1} = b \quad \text{also} \quad a^{-1} = a$$

Therefore $$ba = ab, \ \forall\, a, b \in G.$$

Hence G is an abelian group.

2. If $\forall\, a, b \in G$

$$(ab)^2 = a^2 b^2 \text{ then G is abelian group.}$$

$$(ab)^2 = (ab)(ab) = a(ba)b$$

also $$a^2 b^2 = (aa)(bb) = a(ab)b$$

Now $$(ab)^2 = a^2 b^2 \Rightarrow a(ba)b = a(ab)b$$

By left cancellation law $$(ba)b = (ab)b$$

$\Rightarrow$ $$ba = ab \quad \text{[By right cancellation law]}$$

$\Rightarrow$ G is an abelian group.

9.11 Permutation

Suppose S is a non empty finite set having n distinct elements. Then a one-one onto function from S to S is called a permutation of degree n.

Remark :

1. The number of elements in the finite set S is known as the degree of permutation.
2. $\exists$ n ! distinct permutations from S to S.
3. Set of all such permutations is known as permutation set and is denoted by P_n.

   Hence $P_n = \{f \,;\, f : S \to S$ is a one-one onto function and deg (S) = n$\}$

9.11.1 Permutation group

If P_n is a set of permutations and composite of function is a composition defined on set S, then P_n forms a group and is known as permutation group.

If $n \leq 2$, the permutation group is abelian group.

And if $n > 2$, then it is a non abelian group.

9.11.2 Symbol for a Permutation

Let $S = \{a_1, a_2, \ldots, a_n\}$ be a finite set having n distinct elements. If $f : S \to S$ is one-one onto function, then f is a permutation of degree n.

Let $f(a_1) = b_1, f(a_2) = b_2, \ldots f(a_n) = b_n$,

Then f can be expressed as,

$$f = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix}$$

Each element of second row is f-image of the element of the first row lying directly above it.

9.11.3 Product of Two Permutations

$$f = \begin{pmatrix} a_1 \; a_2 \; \dots \; a_n \\ b_1 \; b_2 \; \dots \; b_n \end{pmatrix}$$

$$g = \begin{pmatrix} b_1 \; b_2 \; \dots \; b_n \\ c_1 \; c_2 \; \dots \; c_n \end{pmatrix}$$

then

$$fg = \begin{pmatrix} a_1 \; a_2 \; \dots \; a_n \\ c_1 \; c_2 \; \dots \; c_n \end{pmatrix}$$

For f replaces a_1 by b_1 and then g replaces b_1 by c_1 so that fg replaces a_1 by c_1 similarly for $a_2, a_3 \dots a_n$.

9.11.4 Identity Permutation

If I is a permutation of degree n such that I replaces each element by the element itself. I is called the identity permutation of degree n.

$$I = \begin{pmatrix} a_1 \; a_2 \; \dots \; a_n \\ a_1 \; a_2 \; \dots \; a_n \end{pmatrix}$$

For example, $S = \{1,2,3\}$. *Then* P_3 *is a set of all distinct one-one onto functions from S onto itself.* $P_3 = \{f : f$ *is a permutation of degree 3\} degree of* $P_3 = 3! = 6$.

Hence number of distinct permutations in P_3 *will be 6*

$$P_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}$$

Let

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, f_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, f_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

$$f_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$ And composite of functions is defined as operation of P_3

Now

$$f_1 f_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = f_3$$

.	I	f_1	f_2	f_3	f_4	f_5
I	I	f_1	f_2	f_3	f_4	f_5
f_1	f_1	I	f_3	f_2	f_5	f_4
f_2	f_2	f_4	I	f_5	f_1	f_3
f_3	f_3	f_5	f_1	f_4	I	f_2
f_4	f_4	f_2	f_5	I	f_3	f_1
f_5	f_5	f_3	f_4	f_1	f_2	I

1. Closure property

Obvious from the table,

e.g. $f_2, f_3 \in P_3$

Then $f_2 f_3 = f_5 \in P_3$

Hence P_3 is closed with respect to composite of mapping as Composition

2. Associative property

Obvious from the table,

$f_1, f_2, f_3 \in P_3$

$$f_1 (f_2 f_3) = f_1 f_5 = f_4$$

also $$(f_1 f_2) f_3 = f_3 f_3 = f_4$$

Hence composite of mapping operation is associative on P_3

3. Existence of identity

I is the identity element of P_3

Such that $I f_1 = f_1 = f_1 I$

4. Existence of inverse

$\forall f \in P_3 \; \exists g \in P_3$

Such that $fg = I$

e.g. Inverse of I is I

Inverse of f_1 is f_1

Inverse of f_2 is f_2

Inverse of f_3 is f_4

Inverse of f_4 is f_3

Inverse of f_5 is f_5

Hence every element of P_3 possesses inverse hence P_3 is a group with respect to composite of function as operation.

This group P_3 is known as permutation group of degree 3! i.e. 6.

9.12 Complexes and Subgroups of a Group

Let (G, *) be a group.

Any nonempty subset H of a group (G, *) is called a complex of the group G.

9.12.1 Subgroup

If (G, *) is a group and H is a subset of G.

i.e. $H \subset G$.

And (H, *) is also a group.

Then (H, *) is known as subgroup of G.

Examples

1. (I, +) is a subgroup of (Q, +) also, (I, +) is a subgroup of ($\mathfrak{R}$, +) and ($\mathbb{C}$, +).
2. $(Q_0, \cdot)$ is a subgroup of $(\mathfrak{R}_0, \cdot)$.
3. $(\mathfrak{R}_0, \cdot)$ is a subgroup of $(\mathbb{C}_0, \cdot)$

Remark : $I \subset Q$

(I, +) is a group.

$(Q_0, \cdot)$ is a group.

But (I, +) is not a subgroup of $(Q_0, \cdot)$.

As I is not a group with respect to the same operation defined on Q_0.

If (G, *) is an abelian group and $H \subset G$, then elements of H will satisfy associative and commutative properties as elements of H are also elements of G and (G, *) satisfies associative property and commutative property.

So in order to prove that a non empty subset H of (G, *) is a subgroup of G, it is sufficient to show that,

1) H is closed with respect to *

   i.e. $\forall$ a, b $\in$ H $\Rightarrow$ a * b $\in$ H

2) $\exists$ e $\in$ H, such that a * e = a = e * a

3) $\forall$ a $\in$ H $\exists$ b $\in$ H

Such that a * b = e = b * a.

i.e. (H, *) satisfies closure property, identity element exist in H and for every element $a \in H$, a^{-1} also exist in H.

Theorem 1 : If (G, *) is a group and H is a non empty subset of G.

Then, H is subgroup of G iff

1) $\forall a, b \in H \Rightarrow a * b \in H$ (closure property)

2) $\forall a \in H \Rightarrow a^{-1} \in H$, where a^{-1} is the inverse of a in G.

Proof :

The conditions are necessary

Suppose (H, *) is a subgroup of (G, *) then (H, *) itself is a group and hence H is closed with * operation.

i.e. $\forall a, b \in H \Rightarrow a * b \in H$

and also for every element $a \in H \ \exists \ a^{-1} \in H$.

Such that $a * a^{-1} = e = a^{-1} * a$

The conditions are sufficient

1. Closure property

Since $a \in H, b \in H \Rightarrow a * b \in H$, therefore H is closed with respect to ' * '.

2. Associative property

The elements of H are also elements of G. The composition '* ' in G is associative. Therefore the same composition must also be associative in H.

3. Existence of identity

The identity element of the subgroup is same as identity of the group.

Now $a \in H \Rightarrow a^{-1} \in H$ [From the given condition 2]

also $a \in H, a^{-1} \in H \Rightarrow a * a^{-1} \in H$ [From the given condition 1]

$$\Rightarrow e \in H$$

Therefore, the identity e is an element of H.

4. Existence of inverse

Since $a \in H \Rightarrow a^{-1} \in H$, therefore each element of H possesses inverse. Hence H itself is a group for the composition in G. So H is a subgroup of G.

Theorem 2 : If (G, $*$) is a group and H is a subset of G, then (H, $*$) will be a subgroup of G iff
$\forall a, b \in H \Rightarrow a * b^{-1} \in H$ where b^{-1} is the inverse of b in G.

Proof : Necessary part

If (H, $*$) is a subgroup of (G, $*$) then (H, $*$) itself is a group.

$$b \in H \Rightarrow b^{-1} \in H$$

And if

$$a, b \in H \Rightarrow a, b^{-1} \in H$$

also,

$$a, b^{-1} \in H \Rightarrow a * b^{-1} \in H$$

as

$$a, b \in H \Rightarrow a * b^{-1} \in H$$

Hence

Therefore the condition is necessary.

Sufficiency part

Now it is given that

$$a \in H, b \in H \Rightarrow a * b^{-1} \in H$$

We are to prove that H is a subgroup of G.

1. Existence of identity

We have $a \in H, a \in H \Rightarrow a * a^{-1} \in H$ [By given condition]

$$\Rightarrow e \in H$$

Therefore the identity e is an element of H.

2. Existence of inverse

Let a be any element of H.

then $e \in H, a \in H \Rightarrow e * a^{-1} \in H$

$$\Rightarrow a^{-1} \in H$$

Therefore each element of H possesses inverse.

3. Closure property

Let $a, b \in H$

$\Rightarrow a, b^{-1} \in H$ [As if $b \in H \Rightarrow b^{-1} \in H$]

$$\Rightarrow a * (b^{-1})^{-1} \in H$$

$\Rightarrow a * b \in H$ [As $(b^{-1})^{-1} = b$]

4. Associative property

Elements of H are also the elements of G. The composition ' $*$ ' in G is associative. Therefore it must also be associative in H.

Hence (H, $*$) itself is a group.

Therefore (H, $*$) is a subgroup of (G, $*$).

Remark : 1. If H is a non-empty subset of (G, +) then necessary and sufficient condition for H to be a subgroup of G will be,

$$\forall\, a, b \in H \Rightarrow a + (-b) \in H$$

$$\text{or } a - b \in H$$

2. Theorem **2** gives us a very important characterization of a subgroup. Whenever we are required to show that a non-empty subset H of a group (G, $*$) is a subgroup of G, we would make use of this theorem.

i.e. $\forall\, a, b \in H \Rightarrow a * b^{-1} \in H$.

Theorem on subgroup

Theorem 3

Intersection of two subgroups is a subgroup.

Proof : Let $(H_1, *)$ and $(H_2, *)$ be any two subgroups of (G, $*$). Then $H_1 \cap H_2 \neq \emptyset$, since at least the identity element e is common to both H_1 and H_2.

In order to prove that $H_1 \cap H_2$ is a subgroup, it is sufficient to prove that,

$$a \in H_1 \cap H_2,\ b \in H_1 \cap H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$$

Now $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

also $b \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1 and H_2 are subgroups of G.

Therefore

$$a \in H_1, b \in H_1 \Rightarrow a * b^{-1} \in H_1$$

also $$a \in H_2, b \in H_2 \Rightarrow a * b^{-1} \in H_2$$

Hence $a \in H_1 \cap H_2,\ b \in H_1 \cap H_2 \Rightarrow a * b^{-1} \in H_1 \cap H_2$

Hence $H_1 \cap H_2$ is a subgroup of (G, $*$).

Theorem 4 : Arbitrary intersection of subgroups, i.e. the intersection of any family of subgroups of a group is a subgroup.

Proof : Let $(G, *)$ be a group.

and $H_1, H_2, \ldots\ldots H_n$ are subgroups of G.

Then we have to show that,

$H_1 \cap H_2 \cap \ldots\ldots\ldots \cap H_n$ is also a subgroup of G.

$$H_1 \cap H_2 \cap H_3 \cap \ldots\ldots \cap H_n = \bigcap_{i=1}^{n} H_i$$

Let $a \in \bigcap_{i=1}^{n} H_i$ and $b \in \bigcap_{i=1}^{n} H_i$

$\Rightarrow a \in H_i, \forall\ i \in N$ and $b \in H_i, \forall\ i \in N$

$\Rightarrow a * b^{-1} \in H_i\ \forall\ i \in N$ [As each H_i is a subgroup of G]

$\Rightarrow a * b^{-1} \in \bigcap_{i=1}^{n} H_i$

$\Rightarrow \bigcap_{i=1}^{n} H_i$ is a subgroup of G.

Remark : 1. The union of two subgroups is not necessarily a subgroup.

For example, let G be the additive group of integers.

Then $H_1 = \{\ldots\ldots\ldots -4, -2, 0, 2, 4, 6, \ldots\ldots\}$

and $H_2 = \{\ldots, \ldots -6, -3, 0, 3, 6, 9, \ldots\}$

are two subgroups of G.

Now $H_1 \cup H_2 = \{\ldots\ldots, -6, -4, -3, -2, 0, 2, 3, 4, 6, 8, 9, \ldots\ldots\}$

$2 \in H_1 \cup H_2$, $3 \in H_1 \cup H_2$

But $2 + 3 = 5 \notin H_1 \cup H_2$

Therefore $H_1 \cup H_2$ is not a subgroup of G.

However $H_1 \cap H_2 = \{\ldots\ldots, -12, -6, 0, 6, 12 \ldots\}$ is a subgroup of G.

2. If we take $H_3 = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$

then $H_1 \cup H_3 = H_1$

and H_1 is a subgroup of G.

Hence union of two subgroups is a subgroup iff one is contained in the other.

Remark on subgroups : Let (G, *) be a group then ({e}, *) and (G, *) are always two subgroups of G called **trivial** or **improper** subgroups. A subgroup other than these two subgroups is called a proper subgroup or non trivial subgroup.

Example 9.18 *Let G be a group, for a fixed element of G, Let $G_x = \{a \in G ; ax = xa\}$. Show that G_x is a subgroup of G for all $x \in G$.*

Solution : Let G be a group and $G_x = \{a \in G \; ; \; ax = xa\}$ then we have to show that G_x is a subgroup of G.

It is sufficient to show that,

i) G_x is a subset of G.

ii) $\forall$ a, b $\in G_x \Rightarrow a\,b \in G_x$

iii) $\forall\, x \in G_x \Rightarrow x^{-1} \in G_x$

G_x is a non-empty set as atleast $e \in G_x$ because $a * e = e * a$.

Hence G_x is a non empty set also every element of G_x is an element of G hence $G_x \subset G$.

Now let a, b $\in G_x$, then we have to prove that $a\,b \in G_x$.

Now as $\quad a \in G_x \Rightarrow ax = xa$

$\qquad\quad b \in G_x \Rightarrow bx = xb$

then $\quad (ab)\,x = a\,(bx)$ [By associative property]

$$= a\,(xb) = (ax)\,b = (xa)\,b = x\,(ab)$$

Hence $a\,b \in G_x$.

To show that $x^{-1} \in G_x$

$$ea = a \Rightarrow x^{-1}\,(x\,a) = a$$

$$\Rightarrow \quad x^{-1}\,(a\,x) = a$$

$$\Rightarrow \quad x^{-1}\,(a\,x)\,x^{-1} = a\,x^{-1}$$

$$\Rightarrow \quad (x^{-1}\,a)\,(x\,x^{-1}) = ax^{-1}$$

$$\Rightarrow \quad (x^{-1}\,a)\,e = a\,x^{-1}$$

$$\Rightarrow \quad x^{-1}\,a = a\,x^{-1}$$

Hence G_x is a subgroup of G.

Example 9.19 *I is a group of integers under addition, H is a subset of I consisting of all multiples of a positive integer m : that is*

H = {...., –2m, –m, 0, m, 2m, ...}, show that H is a subgroup of I.

Solution : H is a subset of I to prove that it is subgroup of I, it is sufficient to prove that,

$$\forall\, a, b \in H \Rightarrow a+(-b) \in H$$

Here, $k_1 m \in H,\ k_2 m \in H \Rightarrow k_1 m+(-k_2 m)$

$$= (k_1 - k_2)\, m \in H$$

As $k_1 - k_2$ is an integer, $\forall\, k_1$ and $k_2 \in I$.

Hence $(k_1 - k_2)$ m is integer multiple of m.

and so $(k_1 - k_2)\, m \in H$.

Hence H is a subgroup of I.

9.13 Cosets

Let (G, *) be a group and H be any subgroup of G. Let a be any element of G.

Then the set

$H * a = \{h * a\ ;\ h \in H\}$ is called a right coset of H in G generated by a. Similarly the set $a * H = \{a * h\ ;\ h \in H\}$ is called a left coset of H in G.

$H * a$ and $a * H$ are subsets of G.

Note : If G is an abelian group then $H * a = a * H$ i.e. each right coset of H in G is also a left coset of H in G.

Example

Let G be the additive group of integers

i.e. $$G = \{\ldots,-2,-1,0,1,2,3,\ldots\}$$

Let H be a subgroup of G and

$$H = \{\ldots,-10,-5,0,5,10,15,\ldots\}$$

Let $1 \in G$, then

$$H + 1 = \{\ldots,-9,-4,1,6,11,\ldots\}$$

For $2 \in G$, $\quad H + 2 = \{\ldots, -8,-3,2,7,12,\ldots\}$

For $3 \in H$ $\quad H + 3 = \{\ldots,-7,-2,3,8,13,\ldots\}$

For $4 \in H$ $\quad H + 4 = \{\ldots,-6,-1,4,9,14,\ldots\}$

For $5 \in H$ $\quad H + 5 = \{\ldots,-5,0,5,10,15,\ldots\}$

Now $\quad H = H + 5$

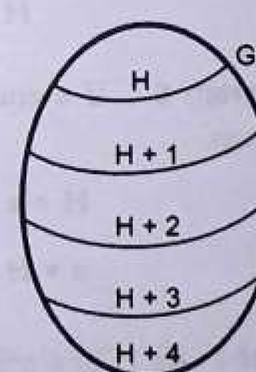


Fig. 9.2

also H, H + 1, H + 2, H + 3, H + 4 are pairwise disjoint sets.

And $H \cup (H + 1) \cup (H + 2) \cup (H + 3) \cup (H + 4) = G$.

Hence, these right cosets of H in G gives partition of G.

This partition is called the right coset decomposition of G.

Theorem 5 : If H is any subgroup of $(G, *)$ and $h \in H$ then $H * h = H = h * H$.

Proof : Let $h \in H$, then to prove that

$$H * h = H$$

Suppose h' is any arbitrary element of H.

Then $h' * h$ is an arbitrary element of $H * h$.

Since H is a subgroup, we have

$$h' \in H, h \in H \Rightarrow h' * h \in H$$

Thus every element of $H * h$ is also an element of H. Hence,

$$H * h \subset H \quad \text{... (1)}$$

Again $\quad h' = h' * (h^{-1} * h) \quad$ [as $h' = h' * e = h' * (h^{-1} * h)$]

$$= (h' * h^{-1}) * h$$

$$\in H * h \quad \text{[as } h' \in H, h^{-1} \in H \Rightarrow h' * h^{-1} \in H]$$

Thus every element h' of H is also an element of $H * h$.

Hence $\quad H \subset H * h \quad$ ... (2)

From equations (1) and (2)

$$H = H * h$$

Similarly we can prove for

$$H = h * H$$

Theorem 6 : If a and b are any two elements of a group $(G, *)$ and H is a subgroup of G, then

$$H * a = H * b \Leftrightarrow a * b^{-1} \in H \text{ and}$$

$$a * H = b * H \Leftrightarrow a^{-1} * b \in H$$

Proof : As H is a subgroup of $(G, *)$

Hence $(H, *)$ itself is a group

$\Rightarrow e \in H$

$\Rightarrow e * a \in H * a$

$\Rightarrow a \in H * a$

If $H * a = H * b$

$\Rightarrow a \in H * b$ [as $a \in H * a$ and $H * a = H * b$]

$\Rightarrow a * b^{-1} \in (H * b) * b^{-1}$

$\Rightarrow a * b^{-1} \in H * (b * b^{-1})$ [Using associative property]

$\Rightarrow a * b^{-1} \in H * e$

$\Rightarrow a * b^{-1} \in H$

Conversely, $a * b^{-1} \in H$

$\Rightarrow H * (a * b^{-1}) = H$ [Using theorem 5, $h \in H$, then $H * h = H$]

$\Rightarrow (H * a) * b^{-1} = H$

$\Rightarrow (H * a) * b^{-1} * b = H * b$

$\Rightarrow (H * a) * e = H * b$

$\Rightarrow H * a = H * b$

Theorem 7 : If a and b are any two elements of a group (G, *) and H is any subgroup of G.
then $a \in H * b \Leftrightarrow H * a = H * b$

Proof : $a \in H * b \Rightarrow a * b^{-1} \in H * b * b^{-1}$

$\Rightarrow a * b^{-1} \in H * e$

$\Rightarrow a * b^{-1} \in H$

Conversely let

$H * a = H * b$

Since $a \in H * a$

Therefore $a \in H * b$

Theorem 8 : Any two right cosets (or left cosets) of H in G are either identical or disjoint.

Proof : Suppose H is a subgroup of $(G, *)$.

$H * a$ and $H * b$ are any two right cosets of H in G.

then we have to prove that either $H * a = H * b$ or $(H * a) \cap (H * b) = \emptyset$.

1) If $(H * a) \cap (H * b) = \emptyset$, nothing to prove.

So let

$c \in H * a \cap H * b$

$\Rightarrow c \in H * a$ and $c \in H * b$

$\Rightarrow c = h_1 * a$ and $c = h_2 * b$

where $h_1, h_2 \in H$

$$\begin{aligned}
\text{Then} \quad h_1 * a &= h_2 * b \\
\Rightarrow h_1^{-1} * (h_1 * a) &= h_1^{-1} * (h_2 * b) \\
\Rightarrow (h_1^{-1} * h_1) * a &= (h_1^{-1} * h_2) * b \\
\Rightarrow \quad e * a &= (h_1^{-1} * h_2) * b \\
\Rightarrow \quad a &= (h_1^{-1} * h_2) * b \qquad [h_1^{-1} * h_2 \in H] \\
\Rightarrow \quad a &\in H * b \\
\Rightarrow \quad H * a &= H * b \qquad \text{[By theorem 7]}
\end{aligned}$$

Hence, two right cosets of H in G are either identical or disjoint.

Remark :

1. i) $a \in H * a$

   ii) $H * a = H * b \Rightarrow a * b^{-1} \in H$

   iii) If $a \in H * b \Leftrightarrow H * a = H * b$

   iv) Either $H * a = H * b$ or $(H * a) \cap (H * b) = \emptyset$.

2. To prove that two right cosets are same (i.e. identical) it is enough to show that one element is common to both and to show that they are non-identical it is enough to show that an element $c \in H * a$ but $c \notin H * b$.

3. The behaviour of right cosets (or left cosets) of H in G is same as equivalence classes, as in equivalence classes we have following properties

   (i) $a \in [a]$

   (ii) $[a] = [b] \Leftrightarrow a R b$ or $(a, b) \in R$

   (iii) if $a \in [b] \Leftrightarrow [a] = [b]$

   (iv) Either $[a] = [b]$ or $[a] \cap [b] = \emptyset$

4. Right cosets of H in G gives partition to group G and equivalence classes give partition to set S.

9.14 Lagrange's Theorem

The order of each subgroup of a finite group is a divisor of the order of the group.

Proof : Let (G, $*$) be a group of finite order n. Number of distinct elements of G is known as order of the group and is denoted by O(G).

Let (H, $*$) a subgroup of G and O (H) = m,

Let $h_1, h_2, \ldots, h_m$ be m distinct elements of H and let $a \in G$. Then $H * a$ is a right coset of H in G.

And $\quad H * a = \{h_1 * a, h_2 * a, \ldots, h_m * a\}$

$H * a$ has m distinct elements,

Since $h_i * a = h_j * a \Rightarrow h_i = h_j$

Therefore each right coset of H in G has m distinct elements. Any two distinct right cosets of H in G are disjoint, i.e. they have no element in common.

Since G is a finite group, the number of distinct right cosets of H in G will be finite, say equal to k.

The union of these k distinct right cosets of H in G is equal to G.

Thus if $H * a_1, H * a_2, \ldots, H * a_k$ are the k distinct right cosets of H in G then

$$G = H * a_1 \cup H * a_2 \cup \ldots \cup H * a_k.$$

$\Rightarrow$ Number of elements in G = Number of elements in $H * a_1$ + Number of elements in $H * a_2$ + ... + Number of elements in $H * a_k$. [As any two right cosets of H in G are disjoint]

$$\Rightarrow \quad O(G) = m + m + m \ldots k \text{ times}$$

$$\Rightarrow \quad O(G) = km$$

$$\Rightarrow \quad \frac{O(G)}{O(H)} = k \qquad \text{where } O(H) = m$$

$\Rightarrow$ O (H) is a divisor of O (G)

$\Rightarrow$ Order of each subgroup of a finite group divides order of the group.

Remark : However the converse of Lagrange's theorem is not true.

If m is a divisor of n, then it is not necessary that G must have a subgroup of order m.

9.15 Integral Powers of an Element of a Group

(Composition denoted by multiplication '·')

Let (G, ·) be a group and let $a \in G$

then by closure property,

$a \cdot a \in G$

i.e. $a^2 \in G$

also $a \cdot a \cdot a$

i.e. $a^3 \in G$

and also

$a \cdot a \cdot a \ldots \cdot a$ (n times)

i.e. $a^n \in G$

a^n is called integral power of a (as n is positive integer)

If e is the identity element of G, then for some a^n of G

$$a^n = e$$

9.16 Integral Multiples of an Element of a Group

(Composition denoted by '+')

Let (G, +) be a group and $a \in G$

Then by closure property

$a + a \in G$

also $a + a + a \in G$

and $a + a + \ldots\ldots + a$ (n times) $\in G$

i.e. $n\,a \in G$

n a is called integral multiple of a (n is positive integer).

9.17 Order of an Element of a Group

Definition : Let (G, ·) be a group.

By the order of an element $a \in G$ is meant the least positive integer n, if one exists, such that

$$a^n = e \text{ (The identity of G)}$$

If there exists no positive integer n such that $a^n = e$, then we say that a is of infinite order or of **zero order.**

Remark : 1. In any group, the identity element e is always of order one and it is the only element of order one.

$$O(e) = 1$$

Also if $O(a) = 1$

Then $a = e$

2. Order of the group G is number of distinct elements in G.

e.g.

1. $(I, +)$ is a group of infinite order.
2. $(\{1, 2, 3, 4\}, X_5)$ is a finite group of order 4.
3. $(\{0, 1, 2, 3, 4, 5, 6, 7\}, +_8)$ is a finite group of order 8.

Example

$G = \{1, -1, i, -i\}$ is a group with multiplication.

1 is the identity element of G.

hence $O(1) = 1$

Now for $O(-1)$

$$(-1)^2 = 1$$

also $(-1)^4 = 1$

$$(-1)^6 = 1$$

But least positive integer n such that

$$(-1)^n = 1 \text{ is } 2$$

Hence $O(-1) = 2$

Similarly $O(i) = 4$

As $i^4 = 1$

And $O(-i) = 4$

As $(-i)^4 = 1$

9.17.1 Some Results on Order of an Element of a Group

1. The order of every element of a finite group is finite and is less than or equal to the order of the group.
2. The order of an element a of a group is same as that of its inverse a^{-1}.

   i.e. $O(a) = O(a^{-1})$
3. Order of any integral power of a can not exceed the order of a.

9.18 Cyclic Group

A group G is called cyclic group if for some $a \in G$, every element $x \in G$ can be written as a^m, where m is some integer, i.e. every element of group G can be generated by some element of G.

Such groups are known as cyclic groups and element a is known as generator of the group G.

And G can be expressed as {a}.

i.e. $$G = \{a\}$$

Remark

A finite group G is cyclic if order of an element of G is the order of the group G.

i.e. $$O(a) = O(G)$$

Example 1 : Cyclic group

Three cube roots of unity forms a multiplicative cyclic group.

$$G = \{1, \omega, \omega^2\}.$$

Such that $\omega^3 = 1$

and $$1+\omega+\omega^2 = 0$$

ω and ω^2 are two generators as all three elements of group G can be formed by only element ω or only element ω^2.

$$\omega, \omega^2, \omega^3 = 1$$

i.e. $$G = \{\omega\}$$

or $$\omega^2, (\omega^2)^2 = \omega, \quad (\omega^2)^3 = 1$$

i.e. $$G = \{\omega^2\}$$

Hence G is a cyclic group.

Example 2 :

Four fourth roots of unity form a cyclic group.

$G = \{1, -1, i, -i\}$.

generators are {i} and {– i}

$$i$$

$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = 1$$

So $$G = \{i\}$$

Also $$-i$$

$$(-i)^2 = -1$$

$$(-i)^3 = i$$

$$(-i)^4 = 1$$

So $$G = \{-i\}$$

Hence G is a cyclic group.

Example 3 :

n, n^{th} roots of unity forms a multiplicative group. This group is cyclic.

$$G = \{1, \omega, \omega^2, \omega^3, \ldots\ldots. \omega^{n-1}\}$$

where $$\omega = e^{\frac{2\pi i}{n}}$$

then each element of G can be expressed as ω^m where m is some integer.

Also, $$\omega^n = \left(e^{\frac{2\pi i}{n}}\right)^n = e^{2\pi i}$$

$$= \cos 2\pi + i \sin 2\pi \quad \text{[Using Euler's result, } e^{i\theta} = \cos\theta + i\sin\theta, \; i = \sqrt{-1}]$$

$$= 1$$

Hence G is a cyclic group.

Result - Every finite group is cyclic.

If ∃ is an element a in the group G.

Such that O (a) = O (G) then a will be the generator of G and G will be a cyclic group.

Hence ({0, 1, 2, ... , n − 1}, '$+_n$')

and ({1, 2, ... , p − 1}, 'X_p') where p is prime are cyclic groups.

$$G = \{0, 1, \ldots\ldots 7\}, '+_8'$$

$$O(1) = 8$$

and $$O(G) = 8$$

Hence 1 is the generator of group G also if 1 is generator its inverse will also be generator.

Hence 7 is also generator.

As

$$7 +_8 7 = 6$$

$$(7 +_8 7) +_8 7 = 6 +_8 7 = 5$$

$$(7 +_8 7 +_8 7) +_8 7 = 5 +_8 7 = 4$$

$$(7 +_8 7 +_8 7 +_8 7) +_8 7 = 4 +_8 7 = 3$$

$$(7 +_8 7 +_8 7 +_8 7 +_8 7) +_8 7 = 3 +_8 7 = 2$$

$$(7 +_8 7 +_8 7 +_8 7 +_8 7 +_8 7) +_8 7 = 2 +_8 7 = 1$$

Also,

$$(7 +_8 7 +_8 7 +_8 7 +_8 7 +_8 7 +_8 7) +_8 7 = 1 +_8 7 = 0$$

Hence every element of G can be generated by 7. Hence 7 is the generator and the group is cyclic group.

9.19 Some Properties of Cyclic Groups

1. Every group of prime order is cyclic.

Proof : **Prime number :** An integer p is said to be a prime number if $p \neq 0$, $p \neq \pm 1$ and the only divisors of p are ± 1, $\pm p$. The least positive prime number is 2.

Suppose G is finite group whose order is a prime number p, then we have to prove that G is a cyclic group.

As least positive prime number is 2, order of G is at least 2.

Hence $\exists\, a \in G$, such that $a \neq e$.

Also $O(a) \geq 2$ as only identity element is of order 1.

Let $O(a) = m$. Then $H = \{a\}$ is a cyclic subgroup of G and $O(H) = O(a) = m$.

By Lagrange's theorem m must be a divisor of p. But p is prime and $m \geq 2$.

Hence $m = p$.

$\therefore H = G$. Since H is cyclic therefore G is cyclic and a is a generator of G.

2. Every cyclic group is an abelian group

Proof : Let $G = \{a\}$ be a cyclic group generated by a. Let x, y be any two elements of G. Then there exist integers r and s such that $x = a^r$, $y = a^s$

Now $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = y x$

Thus $\forall x, y \in G$

$$xy = yx$$

Hence G is an abelian group.

3. From results (1) and (2)

Every group of prime order is cyclic and every cyclic group is abelian. Hence every group of prime order is abelian group.

9.20 Normal Subgroups

Let $(G, *)$ be an abelian group. Let $(H, *)$ be any subgroup of G, then $H * a = a * H, \forall a \in G$ [i.e. each left coset of H in G is a right coset of H in G]

However it is possible that G is not abelian group, yet it possesses a subgroup H such that,

$$H * a = a * H, \forall a \in G.$$

Such subgroup H of group G is known as **normal subgroup** of G.

Normal subgroup

Definition : A subgroup H of a group $(G, *)$ is said to be a normal subgroup of G if $\forall g \in G$ and for $\forall h \in H$.

$$g * h * g^{-1} \in H$$

Remark : Every group G possesses at least two normal subgroups namely G itself and the subgroup consisting of the identity element e alone.

These are called improper normal subgroups. There exist groups for which these are the only normal subgroups, such groups are known as **simple groups.**

Simple group

Definition : A group having no proper normal subgroups is called a simple group.

Every group of prime order is simple group.

Some results on normal subgroups

1. The intersection of any two normal subgroups of a group G is a normal subgroup of G.

   Note : Proof is same as intersection of subgroups.

2. The intersection of any family of normal subgroups is itself a normal subgroup.
3. Every subgroup of abelian group is normal.
4. If H is a subgroup of G and N is normal subgroup of G, then $H \cap N$ is a normal subgroup of G.

9.21 Quotient Groups

Let (G, $*$) be a group.

Let (N, $*$) be a normal subgroup of G.

If $a \in G$, then $N * a$ is right coset of N in G. Also N being normal subgroup of G each right coset of N in G will be a left coset of N in G.

Let G/N be the collection of all cosets of N in G.

i.e. $G/N = \{N * a \,;\, a \in G\}$

G/N is called Quotient set. This quotient set forms a group with respect to composition ' $*$ ' of cosets.

(G/N, $*$) is a group known as quotient group or factor group.

Example 9.20 *(G/N, $*$) is a group.*

Solution : $G/N = \{N * a \,;\, a \in G\}$

1. Closure property

Let $a, b \in G$. Then,

$N * a, N * b \in G/N$

also $(N * a) * (N * b)$

$= N * (a * N) * b$

$= N * (N * a) * b$ [As N is normal subgroup of G $N * a = a * N$]

$= N * N * a * b = N * (a * b)$

$a * b \in G$

Hence $N * (a * b) \in G/N$.

Hence G/N is closed with respect to ' $*$ '.

2. Associative property

Let $a, b, c \in G$

Then $N * a, N * b, N * c \in G/N$

We have $(N * a) * [(N * b) * (N * c)]$

$$= N * a * [N * (b * c)] = N * (a) * (b * c)$$

$$= N * (a * b) * c = (N * (a * b)) * (N * c)$$

$$= ((N * a) * (N * b)) * (N * c)$$

Thus ' * ' in G/N satisfies associative property.

3. Existence of identity

We have $N = N * e \in G/N$.

Also if $N * a \in G/N$

Then $(N * a) * (N * e)$

$$= N * (a * e) = N * a$$

Therefore normal subgroup N itself is the identity of G/N.

4. Existence of inverse

Let $N * a \in G/N$, then $N * a^{-1} \in G/N$ and

$(N * a) * (N * a^{-1})$

$$= N * (a * a^{-1}) = N * e = N$$

Hence $N * a^{-1}$ is the inverse of $N * a$. Thus each element of G/N possesses inverse.

Hence (G/N, *) is a group, known as quotient group.

9.22 Homomorphism of Groups

Let (G, o) be a group and (G′, *) be another group then a mapping $f : (G, o) \rightarrow (G', *)$ is said to be homomorphism if,

$$f(a \text{ o } b) = f(a) * f(b)$$

i.e. combine first the elements a and b of G with operation 'o' and then take the f-image or first take f-images of a and b, i.e. a', b' from G' and combine them by '*' operation, i.e. a * b. If the resultant is same it is known as f **preserves composition** and f mapping is known as **homomorphism** from group (G, o) to group (G′, *) and two groups G and G′ are called homomorphic to each other.

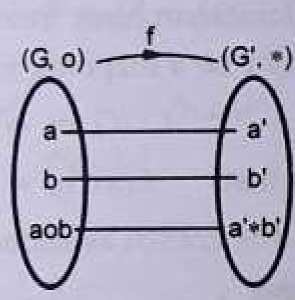


Fig. 9.3

9.22.1 Properties of Group Homomorphism

1. $f(e) = e'$ [Identities corresponds]

2. $f(a^{-1}) = [f(a)]^{-1}$ [Inverse corresponds]

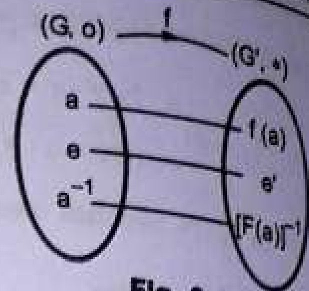


Fig. 9.4

Proof : 1. Let $a \in G$, then $f(a) \in G'$

$$f(a) * e' = f(a) \quad [e' \text{ is the identity of } G']$$

$$= f(a \circ e) \quad [e \text{ is the identity of } G]$$

$$= f(a) * f(e) \quad [f \text{ is homomorphism}]$$

$$\Rightarrow \quad e' = f(e) \quad [\text{by left cancellation law. As } G' \text{ is a group}]$$

2. Let $a \in G$ then $a^{-1} \in G$, as G is a group.

Also $\quad e' = f(e) = f(a \circ a^{-1}) = f(a) * f(a^{-1})$

$$\Rightarrow \quad f(a^{-1}) = [f(a)]^{-1}$$

9.23 Isomorphism of Groups

$f : (G, o) \rightarrow (G', *)$

Let f from group (G, o) to group (G', *) is a mapping.

Such that $f(a \circ b) = f(a) * f(b)$

and also f is one-one onto mapping then f is known as isomorphism from group (G, o) to (G', *) and groups G and G' are known as isomorphic groups, denoted by

$$G \cong G'$$

i.e. a one-one onto homomorphism is known as isomorphism.

9.24 Endomorphism

Homomorphism from group G to itself is called as endomorphism.

Let f be a mapping from (G, o) to itself.

i.e. $\quad f : (G, o) \rightarrow (G, o)$

such that $\quad f(a \circ b) = f(a) \circ f(b)$

then f is known as endomorphism.

9.25 Automorphism of Groups

Isomorphism f from group (G, o) to itself is known as Automorphism.

If f is a mapping from (G, o) to (G, o)

$$f : (G, o) \rightarrow (G, o)$$

Such that

1) $f(a \circ b) = f(a) \circ f(b)$
2) f is one-one onto

Then f is known as automorphism on the group (G, o).

Example 9.21 *Let G be a group with identity e. Show that function $f: G \to G$ defined by $f(a) = e$ for all $a \in G$ is a homomorphism.*

Solution : $f: G \to G$, $f(a) = e$, $\forall a \in G$

We have to prove that f is a homomorphism.

Let $a, b \in G$.

Then $f(ab) = e = e\, e = f(a)\, f(b)$

Therefore f is a homomorphism.

Example 9.22 *Consider the two algebraic systems : $(I, \cdot)$ where I is set of all integers and $\cdot$ is ordinary multiplication operation of integers. $(B, \odot)$ where B is set of all integers and $\odot$ is defined as*

⊙	Positive	Negative	Zero
Positive	Positive	Negative	Zero
Negative	Negative	Positive	Zero
Zero	Zero	Zero	Zero

Show that $(I, \cdot)$ is homomorphic image of $(B, \odot)$.

Solution : Let a and b be any two elements of B.

We have to show that,

if f is a mapping from $(B, \odot)$ to $(I, \cdot)$

i.e. $f: (B, \odot) \to (I, \cdot)$

Then f is a homomorphism. To show that f is a homomorphism from B to I. We have to show that f preserves composition.

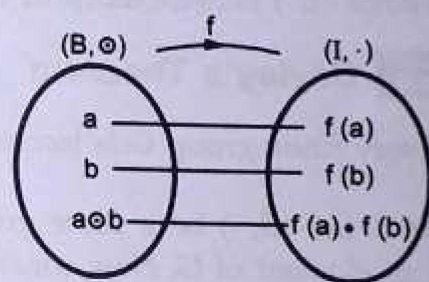


Fig. 9.5

i.e. $f(a \odot b) = f(a) \cdot f(b)$

Let $f(a) = a'$

where a' is positive integer when a is positive, a' is negative integer when a is negative and also $a' = 0$ when a is zero.

Case 1 : If a and b both are positive then $a \odot b$ is positive integer.

Hence,

$$f(a \odot b) = \text{Positive integer} = a' \cdot b'$$

Hence $$f(a \odot b) = f(a) \cdot f(b)$$

Case 2 : If a and b both are negative.

Then $a \odot b$ is positive integer.

Hence $f(a \odot b)$ = Positive integer

$= a' \cdot b'$ [a' and b' are negative but the product is positive]

$= f(a) \cdot f(b)$

Case 3 : a is positive and b is negative.

i.e. $a > 0$, $b < 0$, hence $a \odot b$ is negative then $a' > 0$, $b' < 0$, $a' \cdot b' < 0$

And $f(a \odot b)$

= negative integer

$= a' \cdot b' < 0$

Hence $f(a \odot b) = f(a) \cdot f(b)$

Case 4 : $a = 0$, $b = 0$, then $a' = 0$, $b' = 0$, $a \odot b = 0$

and $f(a \odot b) = 0 = a' \cdot b' = f(a) \cdot f(b)$

Therefore f is a homomorphism from $(B, \odot)$ to $(I, \cdot)$.

Hence $(I, \cdot)$ is homomorphic image of $(B, \odot)$.

9.26 Cayley's Theorem

Every finite group G is isomorphic to a permutation group.

Proof : Let $(G, \cdot)$ be a finite group. If $a \in G$, then for every x in G, the product ax is also an element of G. Now consider the function f_a from $(G, \cdot)$ to $(G, \cdot)$ defined by

$$f_a(x) = ax \ \forall \ x \in G$$

The function f_a is one-one.

As if $x, y \in G$, then

$$f_a(x) = f_a(y) \Rightarrow ax = ay$$

$$\Rightarrow x = y \text{ [By left cancellation law in G]}$$

The function f_a is also onto as if x is any element of G then $\exists$ an element $a^{-1}x \in G$

Such that $f_a(a^{-1}x) = a(a^{-1}x)$

$= (a a^{-1})x = e x = x.$

Thus f_a is a one-one anto function from G to G.

Therefore f_a is a permutation on G.

Let G' denote the set of all such one-one onto functions defined on G corresponding to every element of G.

i.e. $G' = \{f(a) : a \in G\}$

G' form a group with respect to composite of functions as composition.

1. Closure property

Let $f_a, f_b \in G'$ where a, b $\in$ G from definition of product of function.

$$f_a f_b(x) = f_a[f_b(x)] = f_a[bx] = a(bx) = (ab)x$$

$$= f_{ab}(x) \qquad \forall\, a, b \in G$$

Hence $\quad f_a f_b = f_{ab} \qquad$ ...(1)

Also ab $\in$ G then $f_{ab} \in G'$ and thus G' is closed with respect to composite of function.

2. Associative property

Let $f_a, f_b, f_c \in G'$

Then $f_a(f_b f_c) = f_a f_{bc}$ $\qquad$ By equation (1)

$$= f_{a(bc)} = f_{(ab)c} = f_{ab} f_c = (f_a f_b) f_c$$

Therefore composite of functions is associative on G'.

3. Existence of identity

If e $\in$ G, then $f_e \in G'$

And $f_a f_e = f_{ae} = f_a$

Also $f_e f_a = f_{ea} = f_a$

Hence f_e is the identity element of G'.

4. Existence of inverse

$\forall\, a \in G\ \exists\ a^{-1} \in G$,

And hence $f_a \in G'$ and $f_{a^{-1}} \in G'$

Also $\quad f_a f_{a^{-1}} = f_{aa^{-1}} = f_e$

And $\quad f_{a^{-1}} f_a = f_{a^{-1}a} = f_e$

Hence $\quad (f_a)^{-1} = f_{a^{-1}}$

Hence G' is a group.

Now to show that G and G′ are isomorphic, let us define a function

$\phi : G \to G'$

Such that $\phi(a) = f_a$

1. ϕ is one-one

$$\phi(a) = \phi(b)$$

$$\Rightarrow \quad f_a = f_b$$

$$\Rightarrow \quad f_a(x) = f_b(x) \qquad \forall x \in G$$

$$\Rightarrow \quad ax = bx$$

$$\Rightarrow \quad a = b \qquad \text{[By right cancellation law]}$$

2. ϕ is onto

$\forall f_a \in G' \exists a \in G$

Such that $\phi(a) = f(a)$

Hence ϕ is onto.

3. ϕ preserves composition

If $a, b \in G$

$$\phi(ab) = f_{ab} = f_a\, f_b = \phi(a)\ \phi(b)$$

Therefore ϕ preserves composition in G and G′.

hence G is isomorphic to G′

i.e. $\quad G \equiv G'$

9.27 Kernel of Group Homomorphism

Definition : If f is a homomorphism from a group (G, *) into a group (G′, o), then the set K of all those elements of G which are mapped by f on the identity e′ of G′ is called Kernel of homomorphism.

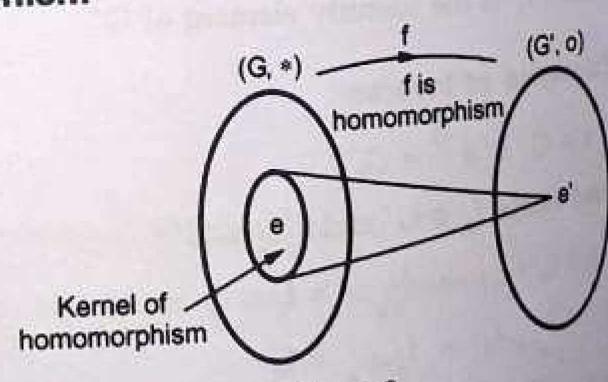


Fig. 9.6

$K = \{a \in G ; f(a) = e'$ where e' is the identity of $G'\}$

where f is a homomorphism from $(G, *)$ to (G', o).

Theorem 9 :

If f is a homomorphism from group $(G, *)$ to group (G', o) with kernel K, then K is a normal subgroup of G.

Proof : Let f be a homomorphism from group $(G, *)$ to group (G', o) and K is kernel of homomorphism f.

then $\quad K = \{a \in G ; f(a) = e'\}$

K is a non-empty set as atleast $e \in K$

Such that $f(e) = e'$

Let $a, b \in K$

$\Rightarrow \quad f(a) = e'$ and $f(b) = e'$

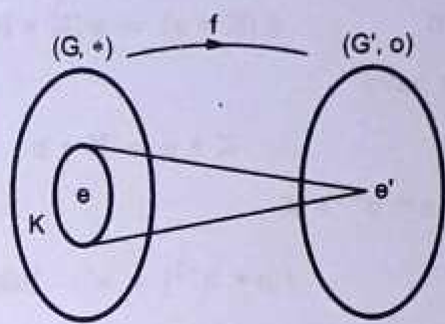


Fig. 9.7

We have,

$$f(a * b^{-1}) = f(a) \text{ o } f(b^{-1}) = f(a) \text{ o } [f(b)]^{-1}$$

$$= e' \text{ o } (e')^{-1} = e' \text{ o } e' = e'$$

$\Rightarrow a * b^{-1} \in K$

$\Rightarrow$ K is a subgroup of $(G, *)$.

Now to prove that K is a normal subgroup of G, we have to show that,

$\forall g \in G,\ k \in K$

$\Rightarrow g * k * g^{-1} \in K$

$$f(g * k * g^{-1}) = f(g) \text{ o } f(k) \text{ o } f(g^{-1}) = f(g) \text{ o } e' \text{ o } f(g^{-1}) = f(g) \text{ o } f(g^{-1})$$

$$= f(g * g^{-1}) = f(e) = e'$$

$\Rightarrow g * k * g^{-1} \in K$

Hence K is a normal subgroup of $(G, *)$.

9.28 Fundamental Theorem on Group Homomorphism

Every homomorphic image of a group G is isomorphic to some quotient group of G.

Proof : Let (G', o) be the homomorphic image of a group $(G, *)$ and f be the corresponding homomorphism. Then f is a homomorphism of G onto G'. Let K be the kernel of this homomorphism. Then K is a normal subgroup of G.

We shall prove that

$$(G/K, *) \cong (G', o)$$

If $a \in G$, then $K * a \in G/K$ and $f(a) \in G'$.

Consider the mapping $\phi : G/K \to G'$.

Such that $\phi(K * a) = f(a), \ \forall \ a \in G$.

First we shall show that the mapping ϕ is well defined, i.e. if $a, b \in G$ and

$$K * a = K * b$$

then $$\phi(K * a) = \phi(K * b)$$

We have

$$K * a = K * b$$

$\Rightarrow \ a * b^{-1} \in K$

$\Rightarrow \quad f(a * b^{-1}) = e' \quad$ [Identity of G′]

$\Rightarrow \quad f(a) \, o \, f(b^{-1}) = e'$

$\Rightarrow \quad f(a) \, o[f(b)]^{-1} = e'$

$\Rightarrow \ f(a) \, o \, [f(b)]^{-1} o \, f(b) = e' \, o \, f(b)$

$\Rightarrow \quad f(a) \, o \, e' = f(b)$

$\Rightarrow \quad f(a) = f(b)$

$\Rightarrow \quad \phi(k * a) = \phi(k * b)$

Therefore ϕ is well defined.

ϕ is one-one

We have,

$$\phi(K * a) = \phi(K * b) \Rightarrow f(a) = f(b)$$

$\Rightarrow \quad f(a) \, o \, [f(b)]^{-1} = f(b) \, o \, [f(b)]^{-1}$

$\Rightarrow \quad f(a) \, o \, f(b^{-1}) = f(b) \, o \, f(b^{-1})$

$\Rightarrow \quad f(a * b^{-1}) = f(b * b^{-1})$

$\Rightarrow \quad f(a * b^{-1}) = f(e)$

$\Rightarrow \quad f(a * b^{-1}) = e'$

$$a * b^{-1} \in K$$

$\Rightarrow$ $$K * a = K * b$$

$\Rightarrow$

$\therefore$ ϕ is one-one.

ϕ is onto G' Let x be any element of (G', o). Then $x = f(a)$ for some $a \in G$ because f is onto G'. Now $K * a \in G/K$ and we have,

$$\phi(K * a) = f(a) = x$$

$\therefore$ ϕ is onto G'.

Finally we have,

$\phi[(K * a) * (K * b)]$

$$= \phi[K * a * b] = f(a * b)$$

$$= f(a) \circ f(b) = \phi(K * a) \circ \phi(K * b)$$

Therefore ϕ is an isomorphism from G/K onto G'.

Hence $$(G/K, *) \cong (G', o)$$

9.29 Rings

Suppose R is a non-empty set equipped with two binary operation called addition and multiplication and denoted by '+' and '·' respectively.

Then the algebraic structure $(R, +, \cdot)$ is known as ring if the following axioms (postulates) are satisfied.

1) **(R, +) is an abelian group**

i.e. closure property for '+'.

i) $\forall$ a, b $\in$ R $\Rightarrow$ $a + b \in R$

ii) Associative property for '+'

$\forall$ a, b, c $\in$ R

$$a + (b + c) = (a + b) + c$$

iii) Existence of identity element for '+'

$\forall$ a $\in$ R $\exists$ 0 $\in$ R

Such that $$a + 0 = a = 0 + a$$

0 is known as additive identity element or zero element of the ring.

iv) Existence of inverse element for '+'

$\forall\ a \in R\ \exists -a \in R$

Such that $a + (-a) = 0 = -a + a$

$-a$ is known as additive inverse of a.

2) (R, ·) is a semi group.

i) Closure property for '·'

$\forall\ a, b \in R \Rightarrow a \cdot b \in R$

ii) Associative property for '·'

$\forall\ a, b, c \in R$

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3) Multiplication '·' distributes over addition '+' from left and also from right.

$\forall\ a, b, c \in R$

i) $a \cdot (b + c) = a \cdot b + a \cdot c$

ii) $(a + b) \cdot c = a \cdot c + b \cdot c$

Example 1 : (I, +, ·) is a ring

as 1) (I, +) is an abelian group.

2) (I, ·) is a semi group.

3) Multiplication distributes over addition.

i) $2 \cdot (3 + 4) = 2 \cdot 3 + 2 \cdot 4$

Also $(3 + 5) \cdot 2 = 3 \cdot 2 + 5 \cdot 2$

Example 2 : (Q, +, ·) is a ring.

Example 3 : ($\Re$, +, ·) is a ring.

Example 4 : ($\mathbb{C}$, +, ·) is a ring.

9.30 Commutative Ring

Ring (R, +, ·) is a commutative ring,

if $\forall\ a, b \in R,\ a \cdot b = b \cdot a$

Example (I, +, ·)

9.31 Ring with Unity

Ring $(R, +, \cdot)$ is known as ring with unity if $\forall\, a \in R\ \exists\ 1 \in R$

Such that $a \cdot 1 = a = 1 \cdot a$

Example $(I, +, \cdot)$

9.32 Commutative Ring with Unity

Ring $(R, +, \cdot)$ is commutative ring with unity,

if $\forall\, a, b \in R$

$$a \cdot b = b \cdot a$$

And $\forall\, a \in R\ \exists\ 1 \in R$

Such that $a \cdot 1 = a = 1 \cdot a$

Example 9.23 *Set M of 2×2 matrices where entries of matrices are real numbers form a ring with unity with respect to addition (+) and multiplication (·) of matrices.*

Solution : 1. Let A and B ∈ M.

Then $A + B \in M$ and $A \cdot B \in M$.

So M is closed with respect to addition and multiplication of matrices.

2. A, B, C ∈ M.

Then $A + (B + C) = (A + B) + C$

Also $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

Hence addition '+' and multiplication '.' operations are associative on M.

3. Existence of identity for addition.

$$\exists \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \in M$$

Such that $A + 0 = A = 0 + A$

4. $$\exists \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \in M$$

Such that $A \cdot I = A = I \cdot A$

I is multiplicative identity element of M.

5. Existence of additive inverse

$\forall\, A \in M\ \exists\ B \in M$

Such that $A + B = 0 = B + A$

6. Commutative property for addition

$\forall$ A, B $\in$ M

$$A + B = B + A$$

7. Distributive laws

$\forall$ A, B, C $\in$ M

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$(A + B) \cdot C = A \cdot C + B \cdot C$$

Hence (M, +, ·) is a ring with unity.

9.33 Properties of a Ring

If R is a ring, then for all a, b, c $\in$ R.

i) $a \cdot 0 = 0 \cdot a = 0$

ii) $a(-b) = -(ab) = (-a)b$

iii) $(-a)(-b) = ab$

iv) $a(b - c) = ab - ac$

v) $(b - c)a = ba - ca$

9.33.1 Without Zero Divisors

If a and b are two elements of set S, then S is without zero divisors. If,

$$a \cdot b = 0 \Rightarrow a = 0 \text{ or } b = 0$$

or $\quad a \neq 0, b \neq 0 \Rightarrow a \cdot b \neq 0$

In integers, product of two integers is zero if one of them is zero hence set of integers is without zero divisors.

9.33.2 With Zero Divisors

If $a \neq 0$, $b \neq 0$ but $a \cdot b = 0$ the set is known as with zero divisors.

e.g. $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$a \neq 0, \quad b \neq 0$

But $ab = 0$

$\Rightarrow$ Matrices are with zero divisors.

9.34 Integral Domain

A commutative ring without zero divisors is an integral domain.

(R, +, ·) is an integral domain, if

1) (R, +) is an abelian group.

2) (R, ·) semi group with commutative property.

3) Multiplication distributes over addition from the left and also from the right.

4) a, b ∈ R

$$a \cdot b = 0 \Rightarrow a = 0 \text{ or } b = 0.$$

Examples

1. (I, +, ·)
2. (2I, +, ·)
3. (Q, +, ·)
4. (ℜ, +, ·)

9.35 Field

A commutative ring with unity in which every non zero element possesses their multiplicative inverse is known as field.

i.e. (F, +, ·) is a field,

if 1) (F, +) is an abelian group.

2) (F_0, ·) is an abelian group.

3) Multiplication distributes over addition from the left and also from the right.

Examples

1. (Q, +, ·)
2. (ℜ, +, ·)
3. (ℂ, +, ·)

are examples of fields.

Remark

If (F, +, ·) is a field then F contains at least two elements, zero element and unit element i.e. additive identity element and multiplicative identity element.

Example 9.24 *Consider the set* $(\mathbb{C}, +, \cdot)$ *where* $\mathbb{C}$ *is the set of complex numbers and + and · are ordinary addition and multiplication operation. Show that* $(\mathbb{C}, +, \cdot)$ *is a ring.*

Solution : Let $\mathbb{C}$ be a non-empty set and '+' and '·' are ordinary addition and multiplication.

1. Closure property for addition '+' and multiplication '·'

$\forall\ z_1, z_2 \in \mathbb{C}$

$\Rightarrow z_1 + z_2 \in \mathbb{C}$

And $z_1 \cdot z_2 \in \mathbb{C}$

where $z_1 = a_1 + i b_1$, $z_2 = a_2 + i b_2$ and a_1, b_1, a_2, b_2 are real numbers and $i = \sqrt{-1}$.

i.e. $\mathbb{C} = \{x \,;\, x = a + i\, b \,;\, a, b \in \Re \text{ and } i = \sqrt{-1}\,\}$

Then $z_1 + z_2 = (a_1 + a_2) + i\,(b_1 + b_2) \in \mathbb{C}$ as $a_1 + a_2$ and $b_1 + b_2$ are real numbers.

Also $\quad z_1 \cdot z_2 = (a_1 a_2 - b_1 b_2) + i\,(a_1 b_2 + a_2 b_1) \in \mathbb{C}$

As $a_1 a_2 - b_1 b_2 \in \Re$

Also $a_1 b_2 + a_2 b_1 \in \Re$

2. Associative property

$\forall\ z_1, z_2, z_3 \in \mathbb{C}$

$\Rightarrow \quad z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

also $\quad z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

hence '+' and '·' are associative compositions on $\mathbb{C}$.

3. Commutative property

$\forall\ z_1, z_2 \in \mathbb{C}$

$\Rightarrow \quad z_1 + z_2 = z_2 + z_1$

Also $\quad z_1 \cdot z_2 = z_2 \cdot z_1$

Hence '+' and '·' are commutative compositions on $\mathbb{C}$.

4. Distributive property

$\forall\ z_1, z_2, z_3 \in \mathbb{C}$

$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

Also $(z_1 + z_2) \cdot z_3 = z_1 \cdot z_3 + z_2 \cdot z_3$

Hence '·' distributes over addition from left and also from right.

5. Existence of identity

$$\forall z = a+ib \in \mathbb{C} \ \exists\ 0+0i \in \mathbb{C}$$

Such that $(a+ib)+(0+0i)$

$$= (a+0)+(b+0)\,i = a+ib$$

hence $0+0\,i$ is the additive identity element of $\mathbb{C}$.

also $\exists\ 1+0\,i$

Such that $(a+ib)\cdot(1+0i) = (a\cdot 1 - b\cdot 0) + (b\cdot i + a\cdot 0)i$

$$= a+bi$$

Hence 1 is the multiplicative identity element of $\mathbb{C}$.

6. Existence of inverse

$$\forall z = a+ib \ \exists\ c+id \in \mathbb{C}$$

Such that $(a+ib)\cdot(c+id) = 1+0\,i$

Then $a\cdot c - b\cdot d = 1$

$$a\cdot d + b\cdot c = 0$$

$$c = \frac{a}{a^2+b^2}, \quad d = -\frac{b}{a^2+b^2}$$

where $c, d \in \Re$ and $c+id \in \mathbb{C}$.

and $\quad c+id = \dfrac{a}{a^2+b^2} - i\,\dfrac{b}{a^2+b^2}$

Hence $(\mathbb{C},+,\cdot)$ is a field.

Example 9.25 *Show that $(F,+,\cdot)$ is a field where F is set of all rational numbers and + and $\cdot$ are ordinary addition and multiplication operators.*

Solution : F is a set of rational numbers and +, $\cdot$ are ordinary addition and multiplication.

Then

1. Closure property for '+' and '·'

$$\forall\ a, b \in F \Rightarrow a+b \in F$$

and $a\cdot b \in F$

Hence F is closed with respect to addition and multiplication.

2. Associative property

$\forall$ a, b, c $\in$ F

$\Rightarrow$ $$a+(b+c) = (a+b)+c$$

And $$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Hence +, $\cdot$ are associative operations on F.

3. Commutative property

$\forall$ a, b, c $\in$ F

$\Rightarrow$ $$a + b = b + a$$

And $$a \cdot b = b \cdot a$$

Hence '+' and '$\cdot$' are commutative on F.

4. Distributive property

$\forall$ a, b, c $\in$ F

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Also $$(a + b) \cdot c = a \cdot c + b \cdot c$$

Hence '$\cdot$' distributes over addition from left and also from right.

5. Existence of identity

$\forall a \in F \; \exists \; 0 \in F$

such that $$a + 0 = a = 0 + a$$

hence 0 is the additive identity element of F.

Also $\exists$ 1 $\in$ F

Such that $$a \cdot 1 = a = 1 \cdot a$$

Hence 1 is the multiplicative identity element of F.

6. Existence of inverse

$\forall a \in F \; \exists \; b \in F$

Such that $$a + b = 0 = b + a$$

Then $$b = -a$$

And b is known as additive inverse.

Also $\exists c \in F$, $c \neq 0$

Such that $$a \cdot c = 1 = c \cdot a$$

Then $$c = \frac{1}{a}$$

Then c is known as multiplicative inverse.

Hence (F, +, ·) is a field.

Theorem 10 : Every field is an integral domain.

Proof : Since (F, +, ·) is a field, F is a commutative ring with unity in which every non zero element possesses multiplicative inverse. To show that (F, +, ·) is an integral domain, we have to show that (F, +, ·) is without zero divisors.

Let $a \cdot b = 0$ and $a \neq 0 \Rightarrow a^{-1}$ exists.

$$\Rightarrow \quad a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0$$

$$\Rightarrow \quad (a^{-1} \cdot a) \cdot b = 0$$

$$\Rightarrow \quad e\,b = 0$$

$$\Rightarrow \quad b = 0$$

Now let $a \cdot b = 0$ and $b \neq 0 \Rightarrow b^{-1}$ exists.

$$\text{so} \quad a \cdot b = 0$$

$$\Rightarrow \quad (a \cdot b) \cdot b^{-1} = 0 \cdot b^{-1}$$

$$\Rightarrow \quad a \cdot e = 0$$

$$\Rightarrow \quad a = 0$$

Thus in a field $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

Therefore a field is without zero divisors and hence is an integral domain.

Remark : Converse is not true. Every integral domain is not a field.

Example : (I, +, ·) is an integral domain but (I, +, ·) is not a field as except 1 and – 1 no element is inversible.

Theorem 11 : Every finite integral domain is a field.

Proof : Let S be a finite set with binary operations '+' and multiplication then being an integral domain (S, +, ·) is a commutative ring without zero divisors.

Let $S = \{a_1, a_2, \ldots, a_n\}$ be n distinct elements of S.

In order to prove that S is a field we have to show that,

1) $1 \in S$

i.e. $\forall\, a \in S$

$$a \cdot 1 = a = 1 \cdot a$$

2) $\forall\, a \in S,\ a \neq 0\ \ \exists\, b \in S$

Such that $a \cdot b = 1$

Let $a \neq 0 \in S$

Consider the n products.

$a \cdot a_1, a \cdot a_2, \ldots\ldots, a \cdot a_n$

All these are elements of S as S is closed with respect to '·' also all these elements are distinct.

For suppose

$$a \cdot a_i = a \cdot a_j$$

$\Rightarrow\ a \cdot (a_i - a_j) = 0$

$\Rightarrow\ a = 0$ or $a_i - a_j = 0$

as S is without zero divisors.

But $a \neq 0 \Rightarrow a_i = a_j$

But all $a_1, a_2, \ldots\ldots, a_n$ are distinct.

So $a_i \neq a_j$ hence $a \cdot a_i \neq a \cdot a_j$.

Hence $a \cdot a_1, a \cdot a_2, \ldots\ldots, a \cdot a_n$ are all the n distinct elements of S placed in some order.

Hence one of these elements will be a. Thus $\exists$ an element, $1 \in S$.

Such that $a \cdot 1 = 1 = 1 \cdot a$

Now we have to show that 1 is multiplicative identity.

Let $b \in S$. Then $\exists\, c \in S$

such that $a \cdot c = b$

$$1 \cdot b = 1 \cdot (a \cdot c) = (1 \cdot a) \cdot c = a \cdot c = b = b \cdot 1$$

Hence 1 is multiplicative identity.

Now as $1 \in s$,

from $a \cdot a_1, a \cdot a_2, \ldots\ldots, a \cdot a_n$ one element will be 1.

Let $a \cdot b = 1 = b \cdot a$ as [S is commutative ring]

$\Rightarrow\ b = a^{-1}$ where $a \neq 0$

Hence inverse exists.

$\Rightarrow$ Every finite integral domain is a field.

Example 9.26 *Show that $R = \{a + b\sqrt{2}; a, b \in I\}$ for the operation +, × is an integral domain but not a field.*

Solution : Let $R = \{a + b\sqrt{2} \; ; \; a, b \in I\}$.

Let $a_1 + b_1\sqrt{2} \in R$ and $a_2 + b_2\sqrt{2} \in R$. Then,

$a_1, b_1, a_2, b_2 \in I$.

We have $(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})$

$= (a_1 + a_2) + (b_1 + b_2)\sqrt{2} \in R$

Since $a_1 + a_2 \in I, \; b_1 + b_2 \in I$.

Also $(a_1 + b_1\sqrt{2}) \times (a_2 + b_2\sqrt{2}) = (a_1 \times a_2 + 2\,b_1 \times b_2) + (a_1 \times b_2 + a_2 \times b_1)\sqrt{2} \in R$

Since $a_1 \times a_2 + 2\,b_1 \times b_2, \; a_1 \times b_2 + a_2 \times b_1 \in I$.

Thus R is closed with respect to addition and multiplication.

All the elements of R are real numbers and we know that addition and multiplication are both associative as well as commutative compositions in the set of real numbers.

Further we have $0 + 0\sqrt{2} \in R$ since $0 \in I$.

If $a + b\sqrt{2} \in R$, then

$(0 + 0\sqrt{2}) + (a + b\sqrt{2}) = (0 + a) + (0 + b)\sqrt{2} = a + b\sqrt{2}$

$\therefore$ $(0 + 0\sqrt{2})$ is the **additive identity.**

Again if $a + b\sqrt{2} \in R$, then $(-a) + (-b)\sqrt{2} \in R$ and we have

$[(-a) + (-b)\sqrt{2}] + (a + b\sqrt{2}) = 0 + 0\sqrt{2}$ therefore each element of R possesses **additive inverse.**

Further in the set of real numbers **multiplication is distributive over addition.**

Again $1 + 0\sqrt{2} \in R$ and we have

$(1 + 0\sqrt{2}) \times (a + b\sqrt{2}) = a + b\sqrt{2} = (a + b\sqrt{2}) \times (1 + 0\sqrt{2})$

Therefore $(1 + 0\sqrt{2})$ is the multiplicative identity.

Thus R is commutative ring with unity.

Now in order to prove that this is an integral domain, we have to show that R is without zero divisors.

Let $(a + b\sqrt{2})$ and $(c + d\sqrt{2})$ be any two elements of ring R. Then

$(a + b\sqrt{2}) \times (c + d\sqrt{2}) = (0 + 0\sqrt{2})$

$\Rightarrow$ $a \times c + 2b \times d = 0$ and $b \times c + a \times d = 0$ and this will happen only when either,

$$a = 0 \text{ and } b = 0$$

or

$$c = 0 \text{ and } d = 0$$

Thus $(a + b\sqrt{2}) \times (c + d\sqrt{2}) = 0 + 0\sqrt{2}$

$\Rightarrow$ Either $a + b\sqrt{2} = 0$ or $c + d\sqrt{2} = 0$

Thus given ring is without zero divisors.

Therefore it is an integral domain.

But it is not a field.

e.g. $3 + 9\sqrt{2}$ is a non-zero element of the ring R. Its inverse would have been

$$\frac{1}{3+9\sqrt{2}} = \frac{3-9\sqrt{2}}{(3+9\sqrt{2})(3-9\sqrt{2})} = \frac{3-9\sqrt{2}}{-153} = \frac{-3}{153} + \frac{9}{153}\sqrt{2}$$

Which is not an element of the ring as $-\frac{3}{153} \notin I$.

Example

$\{Z_5, +_5, \times_5\}$ is an example of finite integral domain which is a field.

$$Z_5 = \{0, 1, 2, 3, 4\}$$

$+_5$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

1) $(Z_5, +_5)$ is an abelian group.
2) $((Z_5)_0, \times_5)$ is an abelian group.
3) Z_5 is without zero divisors.
4) '$\times_5$' distributes over '$+_5$' from left and also from the right and hence $(Z_5, +_5, \times_5)$ is an integral domain which is also a field.

9.36 Subring

Definition : Let (R, +, ·) be a ring. A non-empty subset S of R is called subring of R if (S, +, ·) itself is a ring.

Example : 1) (Q, +, ·) is a ring.

$I \subset Q$

and (I, +, ·) is also a ring.

Hence (I, +, ·) is a subring of (Q, +, ·).

2) ($\mathbb{C}$, +, ·) is a ring

$I \subset \mathbb{C}$, $Q \subset \mathbb{C}$, $\Re \subset \mathbb{C}$

also (I, +, ·) , (Q, +, ·) and ($\Re$, +, ·) are rings.

Hence (I, +, ·) is a subring of ($\mathbb{C}$, +, ·)

(Q, +, ·) is a subring of ($\mathbb{C}$, +, ·)

also ($\Re$, +, ·) is a subring of ($\mathbb{C}$, +, ·)

Necessary and sufficient conditions for subrings

Theorem 12 : The necessary and sufficient conditions for a non-empty subset S of ring (R, +, ·) to be a subring of R are

i) $\forall$ a, b $\in$ S $\Rightarrow$ a − b $\in$ S

ii) $\forall$ a, b $\in$ S $\Rightarrow$ a · b $\in$ S.

Proof : **The Condition is necessary**

Suppose (S, +, ·) is a subring of (R, +, ·).

Since (S, +) is an abelian group.

Therefore $\forall$ a, b $\in$ S $\Rightarrow$ a $\in$ S and − b $\in$ S

$\Rightarrow$ a + (− b) $\in$ S

$\Rightarrow$ a − b $\in$ S

Also (S, ·) is a semigroup.

Therefore $\forall$ a, b $\in$ S $\Rightarrow$ a · b $\in$ S

Hence the conditions are necessary.

The conditions are sufficient

Suppose S is a non-empty subset of R and the conditions (i) and (ii) are satisfied.

From condition (i) we have,

$a \in S, a \in S \Rightarrow a - a \in S$

$\Rightarrow 0 \in S,$ i.e. the additive identity element $\in S$

$0 \in S, a \in S \Rightarrow 0 - a \in S$

$\Rightarrow -a \in S$ i.e. each element of S possesses additive inverse.

Now if

$a, b \in S \Rightarrow a, -b \in S$

$\Rightarrow a - (-b) \in S$

$\Rightarrow a + b \in S.$

$\therefore$ S is closed with respect to addition. Now S is a subset of R. Therefore associative property and commutative property of addition must hold in S since they hold in R.

$\therefore$ (S, +) is an abelian group.

From condition (ii) S is closed with respect to multiplication. Associative property for multiplication and distributive property of multiplication over addition must hold in S since they hold in R.

Hence $(S, +, \cdot)$ is a subring of $(S, +, \cdot)$.

Theorem 13 : Intersection of two subrings is a subring.

Proof : Let $(S_1, +, \cdot)$ and $(S_2, +, \cdot)$ be any two subrings of $(R, +, \cdot)$. Then $S_1 \cap S_2 \neq \emptyset$, since at least the additive identity element 0 is common to both S_1 and S_2.

In order to prove that $S_1 \cap S_2$ is a subring, it is sufficient to prove that

$a \in S_1 \cap S_2, b \in S_1 \cap S_2 \Rightarrow a - b \in S_1 \cap S_2$ and $a \cdot b \in S_1 \cap S_2$

Now $a \in S_1 \cap S_2 \Rightarrow a \in S_1$ and $a \in S_2$

also $b \in S_1 \cap S_2 \Rightarrow b \in S_1$ and $b \in S_2$

But S_1 and S_2 are subrings of R.

Therefore,

$$a \in S_1, b \in S_1 \Rightarrow a - b \in S_1 \text{ and } a \cdot b \in S_1$$

$$\text{also } a \in S_2, b \in S_2 \Rightarrow a - b \in S_2 \text{ and } a \cdot b \in S_2$$

$$\text{Hence } a \in S_1 \cap S_2, b \in S_1 \cap S_2 \Rightarrow a - b \in S_1 \cap S_2 \text{ and } a \cdot b \in S_1 \cap S_2$$

Hence $S_1 \cap S_2$ is a subring of $(R, +, \cdot)$.

Theorem 14 : Arbitrary intersection of subrings, i.e. the intersection of any family of subrings of a ring is a subring.

Proof : Let $(R, +, \cdot)$ be a ring

and $S_1, S_2, \ldots, S_n$ are subrings of R, then we have to show that

$S_1 \cap S_2 \cap \ldots\ldots \cap S_n$ is also a subring of R.

$$S_1 \cap S_2 \cap \ldots\ldots \cap S_n = \bigcap_{i=1}^{n} S_i$$

Let $a \in \bigcap_{i=1}^{n} S_i$ and $b \in \bigcap_{i=1}^{n} S_i$

$\Rightarrow a \in S_i$ and $b \in S_i, \forall\ i \in N$

$\Rightarrow a - b \in S_i$ and $a \cdot b \in S_i, \forall\ i \in N$ [As each S_i is a subring of R]

$\Rightarrow a - b \in \bigcap_{i=1}^{n} S_i$

and $a \cdot b \in \bigcap_{i=1}^{n} S_i$

$\Rightarrow \bigcap_{i=1}^{n} S_i$ is a subring of R.

9.37 Subfields

Definition : Let $(F, +, \cdot)$ be a field. A non-empty subset K of the set F is said to be a subfield of F if $(K, +, \cdot)$ itself is a field.

Examples

1) $(\Re, +, \cdot)$ is a field.

$Q \subset \Re$

And $(Q, +, \cdot)$ is also a field.

Hence $(Q, +, \cdot)$ is a subfield of $(\Re, +, \cdot)$

2) $(\mathbb{C}, +, \cdot)$ is a field.

$Q \subset \mathbb{C},\ \Re \subset \mathbb{C}$

Also $(Q, +, \cdot)$ is a field and $(\Re, +, \cdot)$ is a field.

Hence $(Q, +, \cdot)$ is a subfield of $(\mathbb{C}, +, \cdot)$

And $(\Re, +, \cdot)$ is a subfield of $(\mathbb{C}, +, \cdot)$

Necessary and sufficient conditions for subfield

Theorem 15 : The necessary and sufficient conditions for a non-empty subset K of a field F to be a subfield of F are

i) $a \in K,\ b \in K \Rightarrow a - b \in K$

ii) $a \in K,\ 0 \neq b \in K \Rightarrow a \cdot b^{-1} \in K$

Proof : The conditions are necessary.

Suppose K is a subfield of the field $(F, +, \cdot)$, then $(K, +, \cdot)$ is a field.

Hence i) $(K, +)$ is an abelian group.

ii) $(K_0, \cdot)$ is an abelian group.

iii) multiplication '$\cdot$' distributes over addition.

Now $(K, +)$ is an abelian group,

Hence $b \in K \Rightarrow -b \in K$

Therefore $\forall\, a, b \in K \Rightarrow a - b \in K$.

Also $(K_0, \cdot)$ is an abelian group,

Hence $0 \neq b \in K \Rightarrow b^{-1} \in K$

Therefore $\forall\, a, b \in K \Rightarrow a \cdot b^{-1} \in K$

Hence the conditions are necessary.

The conditions are sufficient

Suppose K is a non-empty subset of field $(F, +, \cdot)$ and conditions (i) and (ii) are satisfied.

To prove that $(K, +, \cdot)$ is a field, we have to show that $(K, +)$ is an abelian group and $(K_0, \cdot)$ is an abelian group also multiplication distributes over addition from the left as well as from the right.

As associative properties for both '+' and '$\cdot$', commutative properties for both '+' and '$\cdot$' and distributive properties are satisfied by the elements of F and hence these properties must hold in K.

Hence we have to prove that

i) $\forall\, a, b \in K \Rightarrow a + b \in K$

ii) $\exists\, 0 \in K$ such that $0 + \ldots$

iii) $\forall\, b \in K\ \exists -b \in K$ such that $a+(-b) = 0 = (-b)+a$

and

iv) $\forall\, a, b \in K_0 \Rightarrow a \cdot b \in K_0$

v) $\exists\, 1 \in K_0$, such that $1 \cdot a = a = a \cdot 1$

vi) $\forall\, b \neq 0,\ b \in K_0\ \exists\, b^{-1} \in K_0$

Such that $b \cdot b^{-1} = 1 = b^{-1} \cdot b$

From condition (i)

$$a \in K, a \in K \Rightarrow a - a \in K$$
$$\Rightarrow 0 \in K$$

Now
$$0 \in K, a \in K \Rightarrow 0 - a \in K$$
$$\Rightarrow -a \in K$$

And
$$\forall\, a \in K, b \in K \Rightarrow a \in K, -b \in K$$
$$\Rightarrow a - (-b) \in K$$
$$\Rightarrow a + b \in K$$

From condition (ii)

$$\forall\, a \in K_0,\ a \in K_0 \Rightarrow a \cdot a^{-1} \in K_0$$
$$\Rightarrow 1 \in K_0$$

Now
$$1 \in K_0, a \in K_0 \Rightarrow 1 \cdot a^{-1} \in K_0$$
$$\Rightarrow a^{-1} \in K_0$$

and $\forall\, a \in K_0,\ b \in K_0 \Rightarrow a \in K_0,\ b^{-1} \in K_0$
$$\Rightarrow a \cdot (b^{-1})^{-1} \in K_0$$
$$\Rightarrow a \cdot b \in K_0$$

Hence all the conditions are satisfied by elements of K, hence (K, +, ·) is a field and being a subset of F, (K, +, ·) is a subfield of (F, +, ·).

9.38 Ideals

Let $(R, +, \cdot)$ be a ring.

Let S be a non-empty subset of R.

S is known as ideal of a ring R if

i) S is a subgroup of R under addition.

i.e. $(S, +)$ is a subgroup of $(R, +)$

ii) $\forall\ s \in S$ and $\forall\ r \in R$

$s \cdot r \in S$ and $r \cdot s \in S$.

Remark : 1. If S is an ideal of a ring R, then S is also a subring of R.

2. To prove that a non-empty subset S of a ring R is an ideal of R, we have to show that,

i) $\forall\ a, b \in S \Rightarrow a - b \in S$

ii) $\forall\ s \in S,\ \forall\ r \in R \Rightarrow s \cdot r \in S$ and $r \cdot s \in S$.

3. Every ring R always possesses two improper ideals : one R itself and the other consisting of 0 only. These are respectively known as the **unit ideal** and the **null ideal**.

Any other ideals of R are called **proper ideals.** A ring having no proper ideals is known as **simple ring.**

Results on ideals

1. Intersection of any two ideals of a ring R is again an ideal of R.
2. An arbitrary intersection of ideals of a ring R is an ideal of R.

Example

1) $(I, +, \cdot)$ is a ring.

2I is a subset of I.

$(2I, +, \cdot)$ is a subring of $(I, +, \cdot)$ also $(2I, +, \cdot)$ is an **ideal** of $(I, +, \cdot)$ as $\forall\ m \in 2I$ $\forall\ n \in I$.

$\Rightarrow m \cdot n \in 2I$.

2) $(Q, +, \cdot)$ is a ring.

$(I, +, \cdot)$ is a subring of $(Q, +, \cdot)$.

But $(I, +, \cdot)$ is not an ideal of $(Q, +, \cdot)$

as $2 \in I, \frac{5}{3} \in Q$

But $2 \cdot \frac{5}{3} \notin I$

3) $(\Re, +, \cdot)$ is a ring, where $\Re$ is set of real numbers.

$(Q, +, \cdot)$ is a subring of $(\Re, +, \cdot)$

But $(Q, +, \cdot)$ is not an ideal of $(\Re, +, \cdot)$

As $\sqrt{7} \in R, \frac{5}{3} \in Q$

But $\frac{5}{3}\sqrt{7} \notin Q$

Similarly $(\Re, +, \cdot)$ is not an ideal of $(\mathbb{C}, +, \cdot)$.

9.39 Quotient Rings or Rings of Residue Classes

Suppose $(R, +, \cdot)$ is a ring and S is an ideal of R. We can form cosets of S in R. Since R is an additive abelian group, therefore if $a \in R$ then right coset $S + a$ of S in R will be left coset $a + S$.

The cosets of S in R are called the residue classes of S in R.

We denote the set of all residue classes of S in R by R/S

$$R/S = \{S + a \; ; a \in R\}$$

R/S is called quotient set or set of cosets of S in R.

R/S form ring with '+' and '·' of cosets and is known as **quotient ring** or **residue class ring.**

Addition in R/S is defined by

$S + a \in R/S, \quad S + b \in R/S$

Then $(S + a) + (S + b) = S + (a + b) \in R/S$

multiplication in R/S is defined by

$$(S + a) \cdot (S + b) = S + a \cdot b$$

9.40 Ring Homomorphism

Let $(R, +, \cdot)$ be a ring and $(R', +', \cdot')$ be another ring then a mapping $f : (R, +, \cdot) \rightarrow (R', +', \cdot')$ is said to be homomorphism if,

$$f(a + b) = f(a) +' f(b)$$

$$f(a \cdot b) = f(a) \cdot' f(b)$$

i.e. 1. Add elements of R and then take the f-image or first take the f images of a and b and then add in R′. Resultent is same.

2. Take $a \cdot b$ and then f-image or first take images of a and b and then take product in R′. Resultent is same.

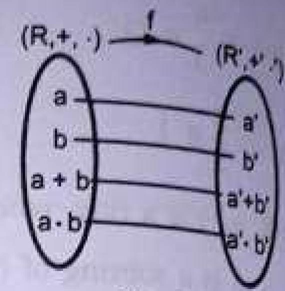


Fig. 9.8

9.41 Ring Isomorphism

If $f : (R, +, \cdot) \to (R', +', \cdot')$

is homomorphism and also one-one onto then f is known as isomorphism and R and R′ rings are known as isomorphic rings, and is denoted by

$$R \cong R'$$

9.42 Kernel of Ring Homomorphism

Definition : If f is a homomorphism of a ring $(R, +, \cdot)$ into a ring $(R', +', \cdot')$, then the set K of all those elements of R which are mapped onto the zero element of R′ is called the kernel of the homomorphism f.

Thus if f is a homomorphism of R into R′, then K is the kernel of f

$$K = \{a \in R \, ; \, f(a) = 0', \text{ where } 0' \text{ is the zero element of } R',$$
$$\text{i.e. additive identity element of } R'\}$$

Result

1) If f is a homomorphism of a ring R into a ring R′ with kernel K, then K is an ideal of R.

2) **Fundamental theorem on homomorphism of rings :**

Every homomorphic image of a ring R is isomorphic to some quotient ring (residue class ring) of R.

i.e. $R/S \cong R'$

Summary

Groups and Rings

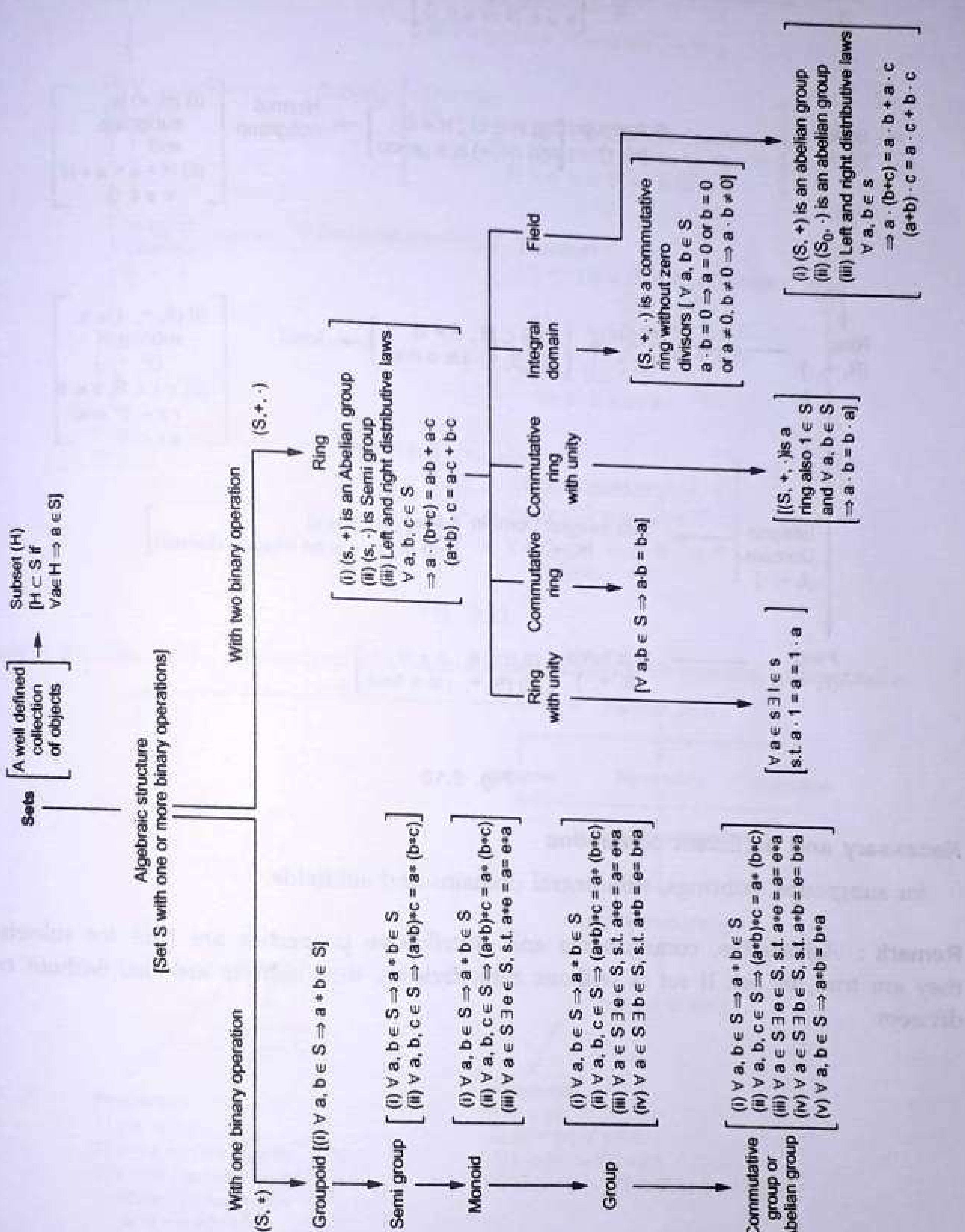


Fig. 9.9

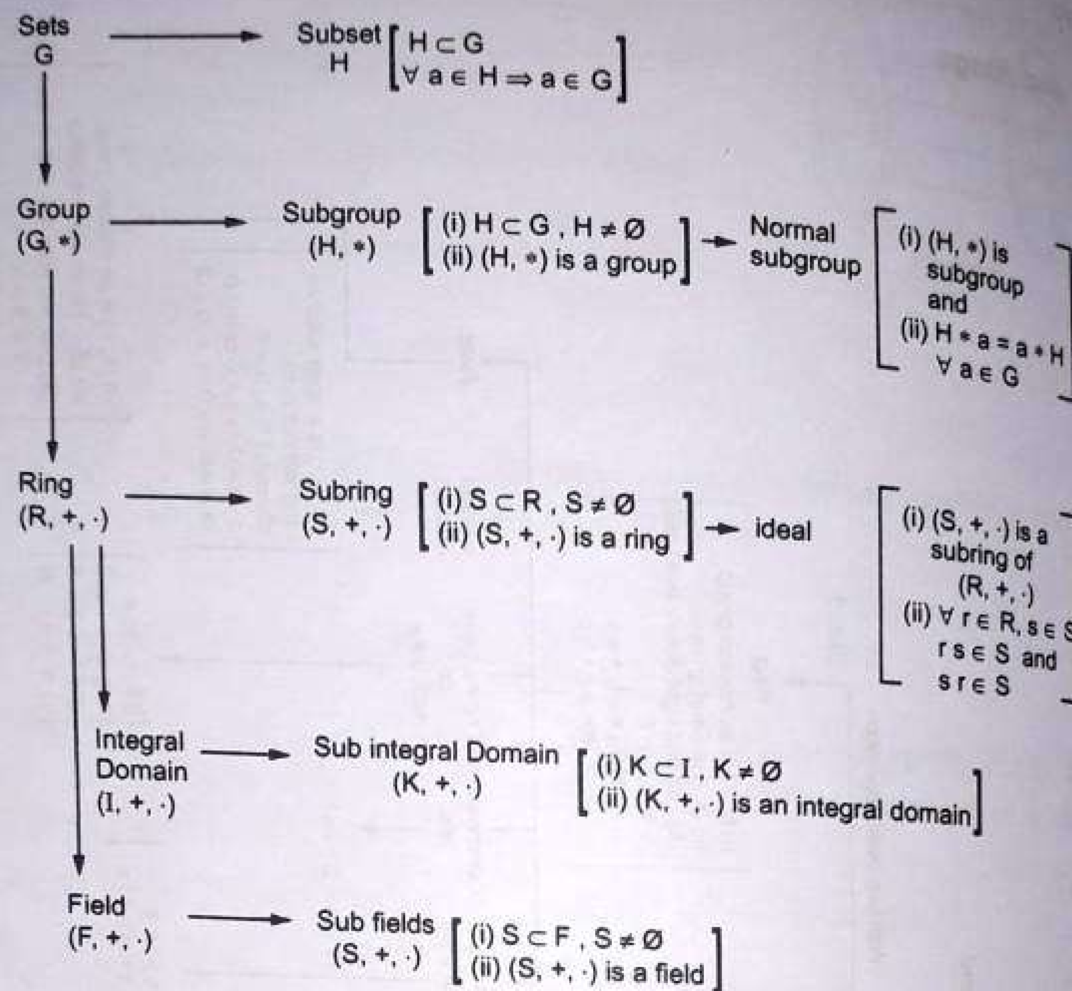


Fig. 9.10

Necessary and Sufficient conditions

for subgroups, subrings, subintegral domains and subfields.

Remark : Associative, commutative and distributive properties are true for subsets if they are true for set. If set is without zero divisors, then subsets are also without zero divisors.

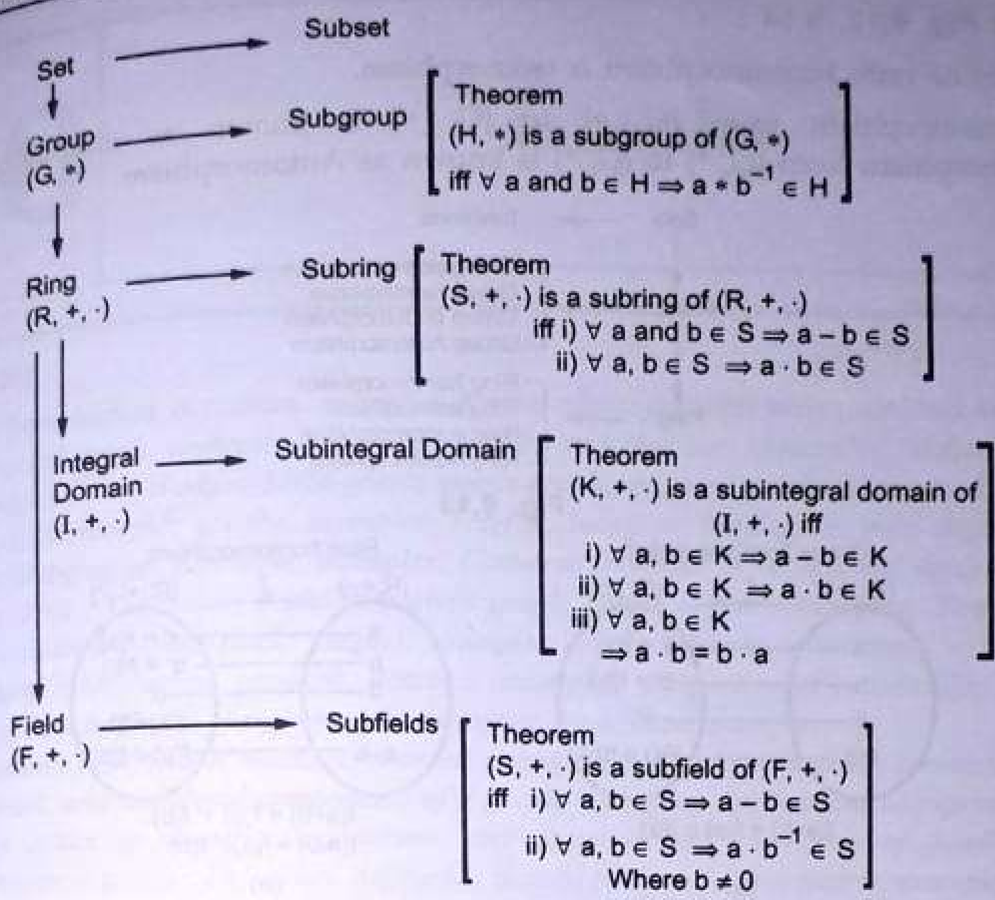


Fig. 9.11

Remark for Fig. 9.12 : Behaviour of cosets and equivalence classes are quite similar.

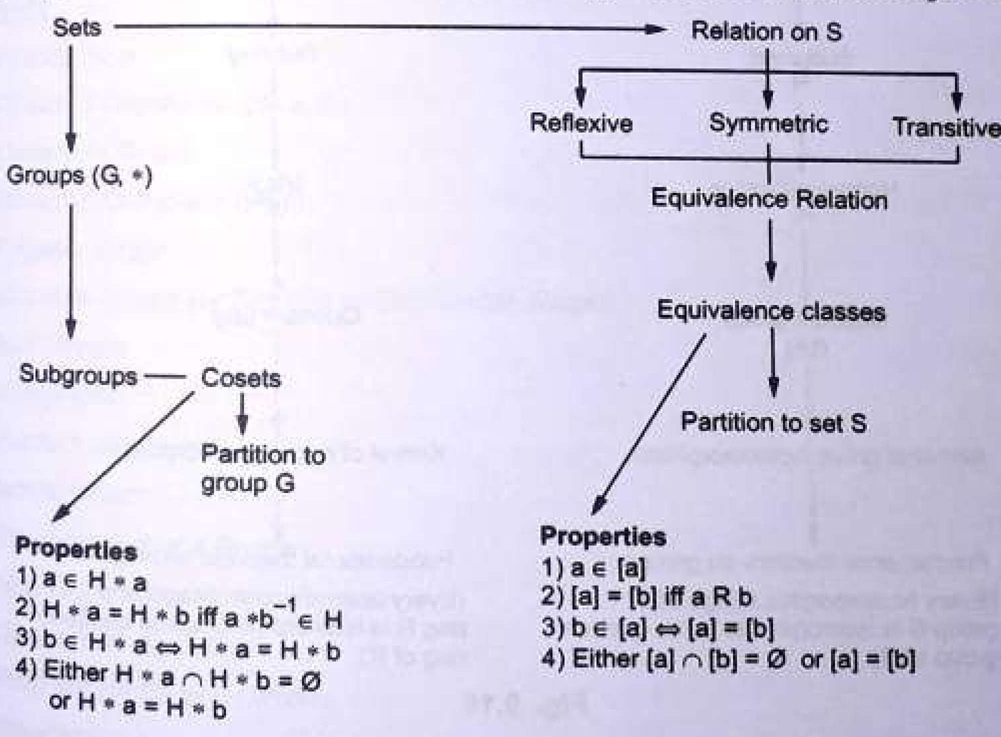


Fig. 9.12

Remark for Fig. 9.13, 9.14 :

1. One-one onto homomorphism is isomorphism.
2. Homomorphism from (G, *) to (G, *) is known as endomophism and isomorphism from (G, *) to (G, *) is known as Automorphism.

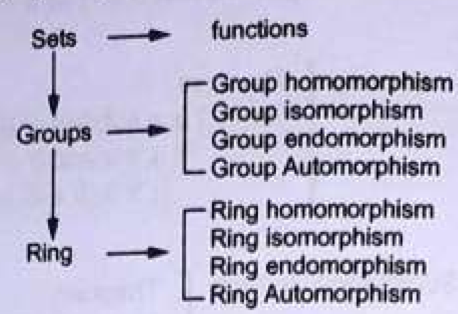


Fig. 9.13

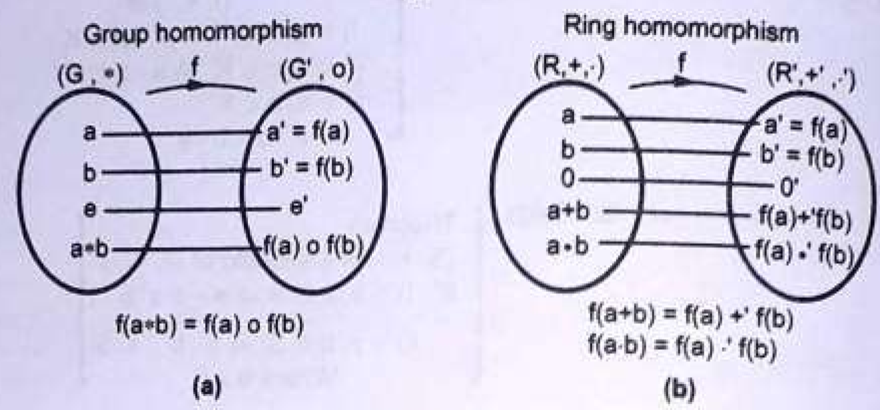


Fig. 9.14

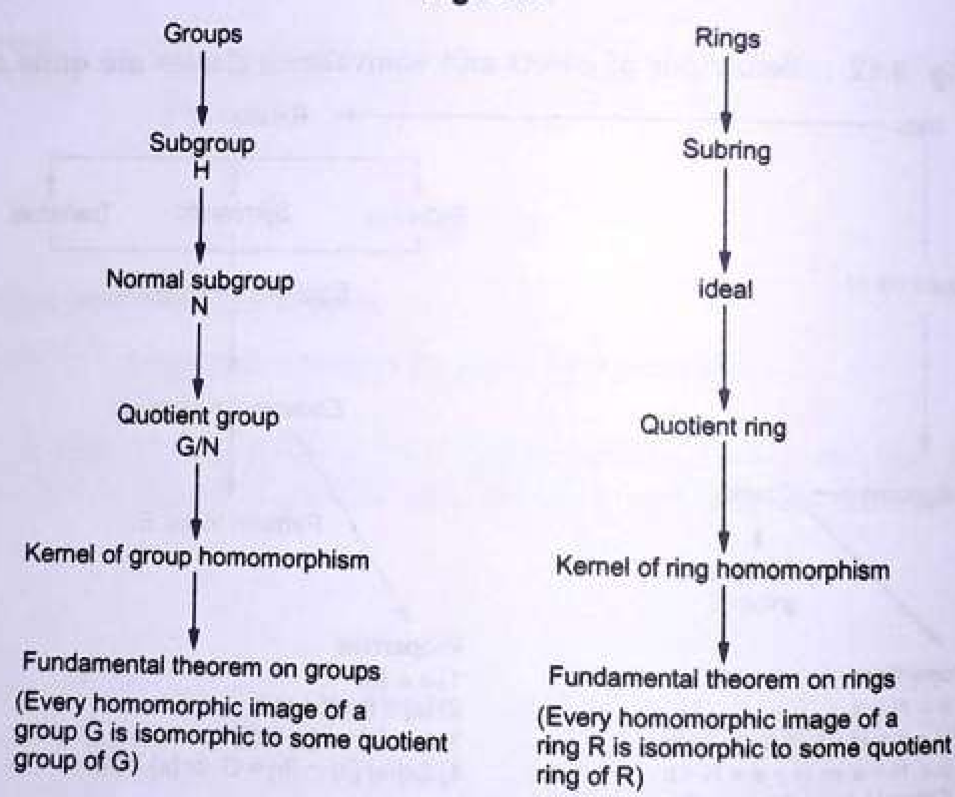


Fig. 9.15

□□□

10 Graphs

Syllabus

***Graphs :** Introduction, definition, examples; Nodes, edges, adjacent nodes, directed and undirected edge, Directed graph, undirected graph, examples; Initiating and terminating nodes, Loop (sling), Distinct edges, Parallel edges, Multi-graph, simple graph, weighted graphs, examples, Isolated nodes, Null graph; Isomorphic graphs, examples; Degree, Indegree, out-degree, total degree of a node, examples; Subgraphs: definition, examples; Converse (reversal or directional dual) of a digraph, examples; Path : Definition, Paths of a given graph, length of path, examples; Simple path (edge simple), elementary path (node simple), examples; Cycle (circuit), elementary cycle, examples;* ***Reachability :** Definition, geodesic, distance, examples; Properties of reachability, the triangle inequality; Reachable set of a given node, examples, Node base, examples;* ***Connectedness :** Definition, weakly connected, strongly connected, unilaterally connected, examples; Strong, weak, and unilateral components of a graph, examples, Applications to represent Resource allocation status of an operating system, and detection and correction of deadlocks; Matrix representation of graph : Definition, Adjacency matrix, boolean (or bit) matrix, examples; Determine number of paths of length n through Adjacency matrix, examples; Path (Reachability) matrix of a graph, examples; Warshall's algorithm to produce Path matrix, Flowchart.*

Contents

10.1 Introduction

There are several reasons for the acceleration of interest in graph theory. There are many applications of graph theory to some areas of physics, chemistry, communication science, computer technology, architecture, operational research, civil engineering, electrical circuits, anthropology, economics, linguistics etc.

The theory is also related to many branches of mathematics including matrix theory, probability, group theory, numerical analysis, combinatorics.

1. Euler (1707 - 1782) became the father of graph theory when in 1736 he settled a famous unsolved problem known as Konigsberg Bridge Problem. There were two islands linked to each other and to the banks of the Pregel river by seven bridges. The problem was to begin at any of the four land areas, walk across each bridge exactly once and return to the starting point.

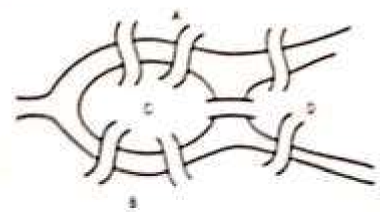


Fig. 10.1 A park in Konigsberg

2. Kirchhoff developed the theory of trees in 1847 in order to solve the system of simultaneous linear equations.
3. In 1859, Sir William Hamilton invented a game. He made a regular dodecahedron whose 20 vertices are labeled with the names of famous cities. The player is challenged to travel around the world by finding a closed circuit along the edges which passes through each vertex exactly once.

10.1.1 Graphs

Let V be the non empty set of points and E be the set of lines between points, then ordered pair (V, E) is known as graph.

The elements of V are also known as point, vertex, node, junction, 0-simplex, and elements of E are known as edge, line, arc, branch, 1-simplex.

V = {a, b, c, d, e, f, g} is set of vertices.

E = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ is set of edges.

e_1 = {a, g} is edge.

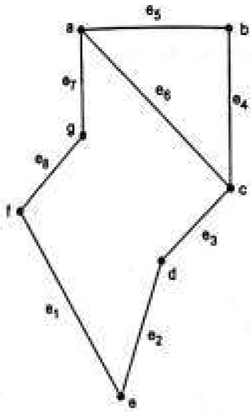


Fig. 10.2

a, g are known as **end vertices** or terminal vertices of the edge e_7. a and g are adjacent vertices. (Some times denoted by a adj g). The vertex a and edge e_7 are incident with each other similarly vertex g and edge e_7 are also incident with each other.

10.1.2 Multigraphs

In a multigraph more than one edge can join two vertices, but no edge can join to itself i.e. self loops are not permitted, but parallel edges are allowed.

e.g.

e_3 and e_4 are parallel edges between vertices a and c.

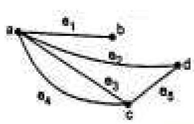


Fig. 10.3

and no self loop is present hence the above graph is a multigraph.

10.1.3 Pseudographs

If self loops as well as parallel edges are allowed, the graph is known as pseudograph.

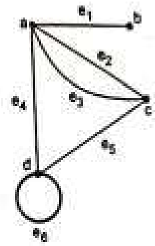


Fig. 10.4

In the above graph e_6 is self loop, e_2 and e_3 are parallel edges between vertices a and c.

Hence the above graph is pseudographs.

10.1.4 Simple Graphs

If no self loop and no parallel edges are present in a graph, the graph is known as simple graph.

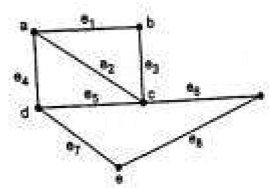


Fig. 10.5

In the above graph no loop is present and also no parallel edge in present hence the graph is known as simple graph.

Remark : In our discussion, we will refer a multigraph or pseudograph as graph and a graph without parallel edges and self loops as simple graphs.

10.1.5 Degree of Vertex

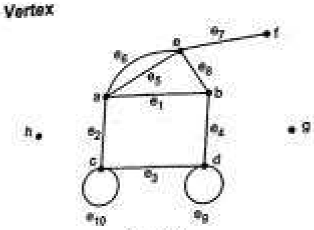


Fig. 10.6

The degree of the vertex a is 4 as e_1, e_2, e_5 and e_6 edges are incident with vertex a.

$e_1 = (a, b),\ e_2 = (a, c),\ e_5 = (a, e),\ e_6 = (a, e)$

While considering degree of b, edges e_1, e_8, e_4 are incident with b.

Hence deg (b) = 3

$e_1 = (b, a),\ e_4 = (b, d),\ e_8 = (b, e)$

Hence the edge is counted in degree of both the terminal vertex.

This way the self loop $e_{10} = (c, c)$ hence e_{10} is counted twice while considering the degree of c.

Therefore degree of c is 4.

edges e_2, e_3, e_{10}, e_{10} are incident with c.

$e_2 = (c, a),\ e_3 = (c, d),\ e_{10} = (c, c).$

Similarly

deg (d) = 4, deg (e) = 4, def (f) = 1.

deg (g) = 0, deg (h) = 0.

A vertex with degree 1 is known as **pendant vertex** hence f is **pendant vertex.**

Also if degree of vertex is zero, the vertex is known as **isolated vertex.**

Vertices g and h are **isolated vertices** as no edge is incident with these vertices.

Theorem 1 : The sum of the degrees of the vertices of a graph is twice the number of edges.

$$\sum \deg v_i = 2e$$

Remark : As every edge is counted in degree of both the terminal vertices hence sum of degree of vertices is twice the number of edges. This is also known as **Handshaking lemma.**

Corollary : In any graph, the number of vertices of odd degree is even.

Proof : Let G be a graph with 'e' edges and 'n' number of vertices. Then by theorem 1.

$$\sum_{i=1}^{n} \deg v_i = 2e$$

Now, the total degree of all the vertices can be expressed as the sum of degrees of even degree vertices and odd degree vertices.

$$\Rightarrow \sum \deg (v_e) + \sum \deg (v_o) = \sum_{i=1}^{n} \deg (v_i)$$

where v_e denotes even degree vertices and v_o denotes odd degree vertices.

$$\Rightarrow \sum \deg v_e + \sum \deg v_o = 2e$$

$$\Rightarrow \sum \deg v_o = 2e - \sum \deg v_e$$

Now sum of degree of even vertices is always even and also 2e is even hence $\sum \deg (v_o)$ is even.

Hence proved.

Results

1. Degree of any vertex of a simple graph with n vertices is at the most (n – 1).
2. Maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Proof : As at the most vertices may be of degree (n – 1) in a simple graph and by theorem 1.

$$\sum \deg v_i = 2e$$

$$\Rightarrow (n-1)+(n-1)+ \ldots\ n \text{ times} = 2e$$

$$\Rightarrow n(n-1) = 2e$$

$$\Rightarrow e = \frac{n(n-1)}{2}$$

Example 10.1 *How many nodes are necessary to construct a graph with exactly 8 edges in which each node is of degree 2.*

Solution : Suppose there are n nodes (vertices) in the graph with 8 edges.

Each vertex is of degree 2.

Then by theorem 1,

$$\sum_{i=1}^{n} \deg (v_i) = 2e$$

$$2 + 2 + 2 \ldots\ n \text{ times} = 2 \times 8$$

$$2n = 2 \times 8$$

$$n = 8$$

Hence 8 vertices are required to construct a graph with 8 edges in which each vertex is of degree 2.

Remark : n nodes are necessary to construct a graph with exactly n edges in which each node is of degree 2.

Example 10.2 *Draw a simple graph with 6 nodes all of degree 2 or greater and with at least 2 nodes of degree 3.*

Solution :

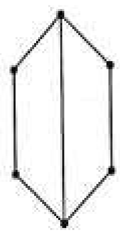


Fig. 10.7

Example 10.3 *Draw a graph with 4 nodes and 7 edges.*

Solution :

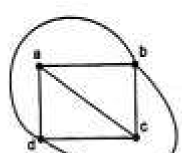


Fig. 10.8

Remark : It is a multigraph as there exist two edges between b and d.

However a **simple graph** with 4 vertices and 7 edges is **not possible** as in a simple graph with n vertices the maximum number of edges will be $\frac{n(n-1)}{2}$.

Hence with 4 vertices, the maximum number of edges will be $\frac{4 \times 3}{2} = 6$.

Example 10.4 *Is there exist a regular graph of degree 5 on 9 vertices ?*

Solution : A regular graph with 9 vertices and degree 5 will have degree sum as

$$\sum \deg(v) = 5 \times 9 = 45$$

Which is an odd number.

But $\sum \deg(v) = 2e$ which is always even hence such type of regular graph is not possible.

Example 10.5 *Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.*

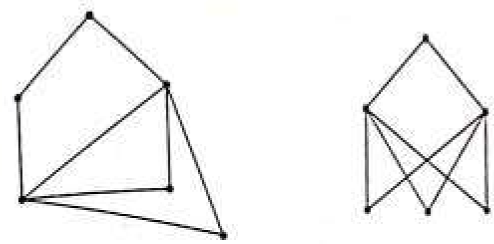
Solution : By theorem 1 (Handshaking lemma)

$$\sum \deg v_i = 2e$$

$$\Rightarrow (2+2+2+2)+(4+4) = 2e$$

(As out of 6 vertices, 4 vertices are of degree 2 and 2 vertices are of degree 4.)

$$\Rightarrow e = 8$$



(a) (b)

Fig. 10.9

Example 10.6 *An odd fellow wishes to have an odd party that is attended by an odd number of odd people, each of whom is acquainted with an odd number of other odd people at the party. Can this odd situation occur ?*

Solution : Represent each person (except host) by a vertex.

Each pair of acquaintance by an edge. Then the odd party can be represented by a graph with odd number of vertices such that the degree of each vertex is odd.

But this is a contradiction to the corollary of theorem 1 which says that number of odd vertices in any graph is always even.

Hence the odd party is not possible.

Example 10.7 *Three married couples on a journey come to a river where they find a boat which cannot carry more than two persons at a time. The crossing of the river is complicated by the fact that the husbands are all very jealous and will not permit their wives to be left without them in a company where there are other man present. Construct a graph to show how the transfer can be made.*

Solution : Let three couples be denoted by $(h_1, w_1), (h_2, w_2)$ and (h_3, w_3).

The possible way to cross the river.

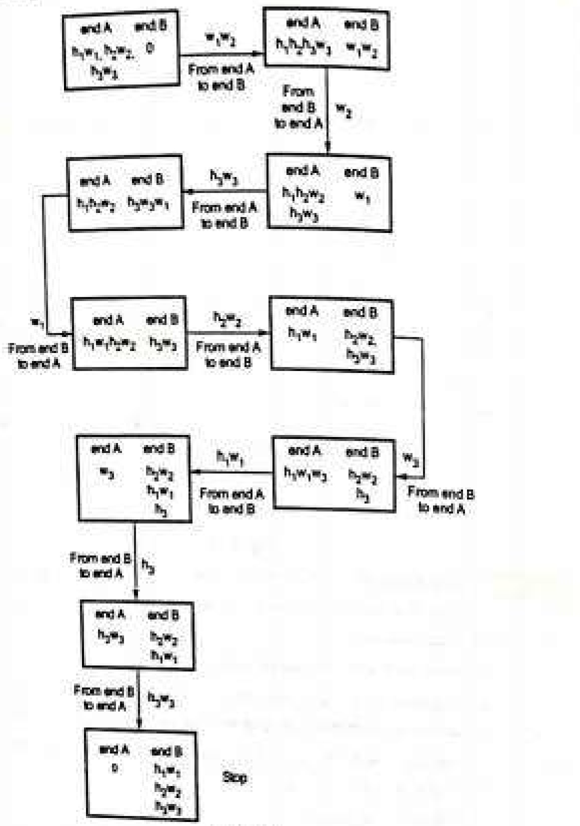


Fig. 10.10

Remark : The puzzle in example 10.7 cannot be solved if there are four couples and the boat can not carry more than two persons at a time. However, the puzzle in example 10.7 can be solved if there are four couples and the boat can not carry more than three persons at a time.

Example 10.8 *A man is supposed to bring a dog, a sheep and a bag of cabbage across a river on a row boat. The boat is very small, and he can carry only one of these items on the boat at a time. Furthermore he cannot leave the dog alone with the sheep nor the sheep alone with the cabbage. Construct the graph and determine all possible ways for the man to transport all items across the river.*

Solution : Let M denotes man, S denotes sheep, D denotes dog and C denotes cabbage.

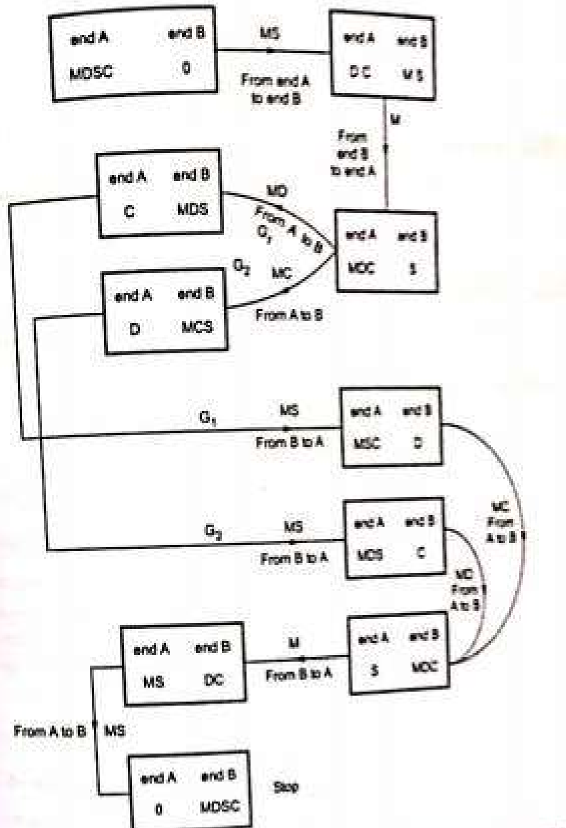


Fig. 10.11

10.1.6 Weighted Graphs

A graph is known as weighted graph if some weight (positive real number) is assigned to each vertices or each edges or to both.

Example :

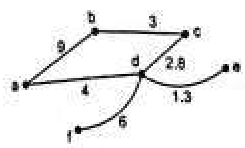


Fig. 10.12

10.1.7 Finite and Infinite Graphs

A graph with finite number of vertices and finite number of edges is known as finite graph.

If a graph is not finite, it is known as infinite graph.

10.2 Directed Graphs or Digraphs

If in a graph each edge has a direction, then the graph is known as directed graph or digraph.

Examples :

1.

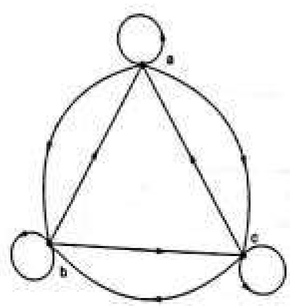


Fig. 10.13

The above graph is a digraph of equivalence relation.

2.

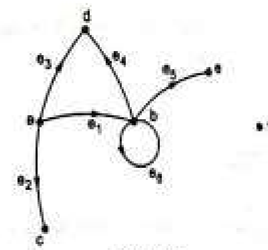


Fig. 10.14

In example 2, Fig. 10.14,

V = {a, b, c, d, e, f} is set of vertices.

E = $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is set of edges.

e_1 denote (a, b).

e_1 is said to be **incident** with vertices a and b.

For e_1, a is known as **initial** vertex and b is known as terminal vertex.

Similarly

e_2 = (a, c), e_3 = (a, d), e_4 = (b, d), e_5 = (b, e), e_6 = (b, b).

e_2 is incident with vertices a and c. In other words it is known as e_2 is incident from vertex a and incident into vertex b.

e_6 is incident from vertex b and incident into vertex b, i.e. e_6 = (b, b).

Such edges are known as **loops** or **self loops**.

10.2.1 Adjacency for Directed Graphs

Two vertices are said to be adjacent if they are joined by an edge.

In example 2, Fig. 10.14 e_4 = (b, d), hence b and d are adjacent vertices. Similarly (a, d) are adjacent vertices also (b, e) and (a, c) but a and e are not adjacent vertices as they are not joined by an edge.

For e_1 = (a, b)

a is known as adjacent to the vertex b and b is known as adjacent from the vertex a.

10.2.2 Indegree and Outdegree

The indegree of a vertex a of a digraph is defined as number of edges which are incident into a and is denoted by,

$\overleftarrow{\deg}(a)$

Similarly, the outdegree of a vertex a of a digraph is defined as number of edges which are incident from a and is denoted by,

$\overrightarrow{\deg}(a)$

In example 2,

$\overrightarrow{\deg}(a) = 0,$ $\overleftarrow{\deg}(a) = 3$

$\overrightarrow{\deg}(b) = 2,$ $\overleftarrow{\deg}(b) = 3$

Remark : For self loop degree is considered in indegree as well as in outdegree.

$\overrightarrow{\deg}(c) = 1,$ $\overleftarrow{\deg}(c) = 0$

$\overrightarrow{\deg}(d) = 2,$ $\overleftarrow{\deg}(d) = 0$

$\overrightarrow{\deg}(e) = 1,$ $\overleftarrow{\deg}(e) = 0$

$\overrightarrow{\deg}(f) = 0,$ $\overleftarrow{\deg}(f) = 0$

10.2.3 Underlying Graph of a Digraph

The underlying graph of a digraph is obtained by neglecting the directions of the edges.

Example :

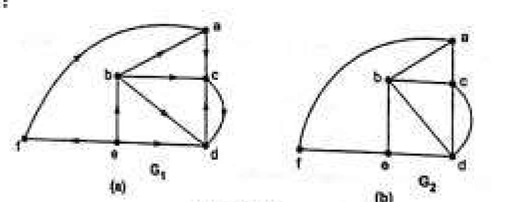


Fig. 10.15

G_2 Fig. 10.15 (b) is underlying graph of G_1, Fig. 10.15 (a).

Remark : G_2 has parallel edges between c and d hence G_2 is a multigraph while G_1 is a simple graph.

10.3 Complete Graph

A simple graph with n vertices is known as complete graph if degree of each vertex is (n − 1), i.e. every pair of vertices is adjacent.

Complete graph with n vertices is denoted by K_n.

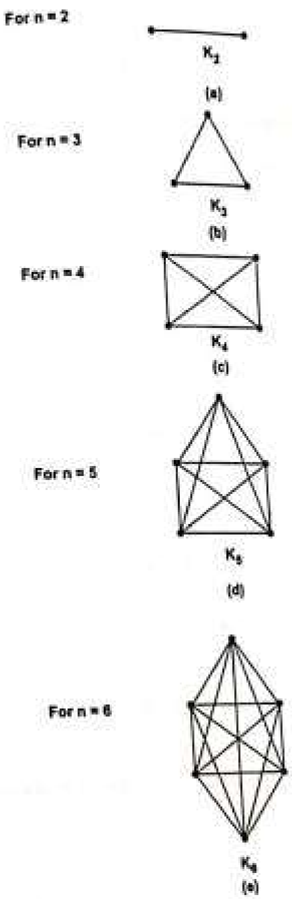


Fig. 10.16

Remark : Number of edges in a complete graph K_n are $\frac{n(n-1)}{2}$

as $\sum \deg(v_i) = 2e$

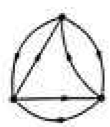
and degree of all n vertices is $n - 1$ hence $e = \frac{n(n-1)}{2}$.

10.4 Directed Complete Graph

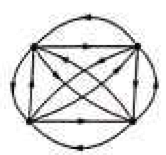
A directed complete graph of n vertices is a graph in which there is exactly one arrow between each pair of distinct vertices.

Example :

(a)



(b)



(c)

Fig. 10.17

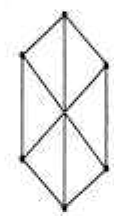
10.5 Regular Graph

In a graph if degree of each vertex is same, then the graph is known as regular graph.

1. Fig. 10.18 (a) is a regular graph as degree of each vertex is 2.

(a)

2. Fig. 10.18 (b) is a regular graph as degree of each vertex is 3.



(b)

3. Fig. 10.18 (c) is a regular graph as degree of each vertex is 2.

This is also a complete graph. Hence every complete graph is regular but the converse is not true. i.e. every regular graph is not complete.

(c)

Fig. 10.18

Hence,

Complete graph $\Rightarrow$ Regular graph

But converse need not be true.

10.6 Bipartite Graph (or Bigraph or Bicolorable Graph)

G is a graph whose set of vertices is V. If V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins V_1 with V_2 also $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$.

Remarks :

1. Vertices of V_1 are not joined.
2. Vertices of V_2 are not joined.
3. Bipartite graph can't have self loop.

Examples :

1.

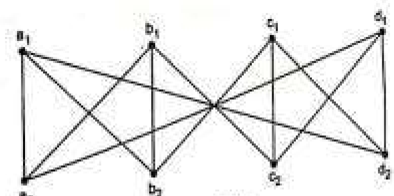


Fig. 10.19

$V_1 = \{a_1, b_1, c_1, d_1\}$

$V_2 = \{a_2, b_2, c_2, d_2\}$

$V_1 \cup V_2 = V, \quad V_1 \cap V_2 = \varnothing$

d vertices of V_1 are joined with vertices of V_2.

ence this is a bipartite graph.

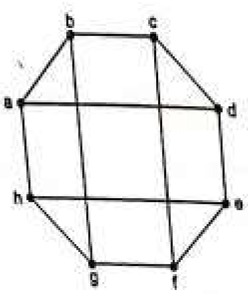


Fig. 10.20

$V = \{a, b, c, d, e, f, g, h\}$

$V_1 = \{a, c, e, g\}$

$V_2 = \{b, d, f, h\}$

then $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \varnothing$.

Also a is joined with b, d and h i.e. vertex of V_1 is joined with vertices of V_2. Similarly b of V_2 is joined with a, g and c of V_1 and no two vertices of V_1 are joined also no two vertices of V_2 are joined.

Hence the above graph is a bipartite graph.

10.6.1 Complete Bipartite Graph

A bipartite graph is known as complete bipartite graph if each vertex of V_1 is joined with every vertex of V_2.

A complete bipartite graph is denoted by $K_{m,n}$ where number of vertices of V_1 are m and number of vertices of V_2 are n. The total number of edges in a complete bipartite graph $K_{m,n}$ is mn.

Examples :

1.

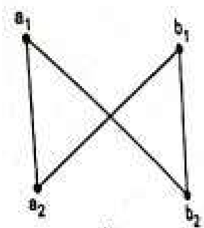


Fig. 10.21

Hence number of edges are 4.

2.

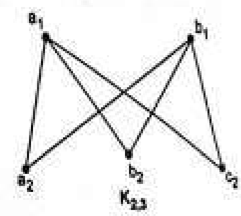


Fig. 10.22 (a)

Number of edges are 6.

3.

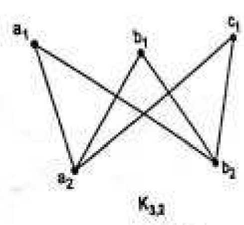


Fig. 10.22 (b)

Number of edges are 6.

4.

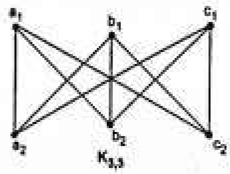


Fig. 10.23

Number of edges are 9.

$K_{3,3}$ is a complete bipartite graph which is also regular as degree of each vertex is 3.

Hence a complete bipartite graph is regular if in $K_{m,n}$, $m = n$.

5.

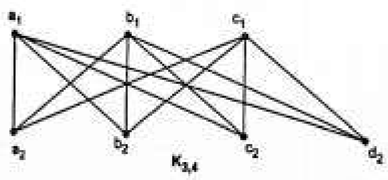


Fig. 10.24

6.

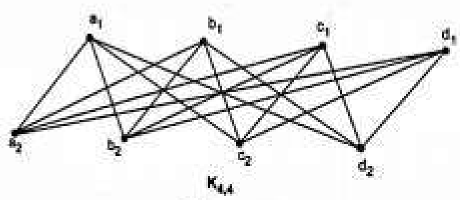


Fig. 10.25

$K_{4,4}$ is complete bipartite graph which is regular.

7.

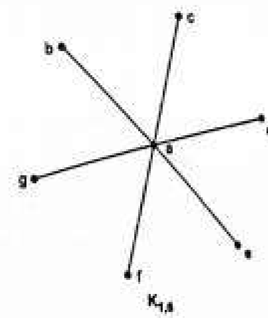


Fig. 10.26

$K_{1,6}$ is a complete bipartite graph, also known as **star**.

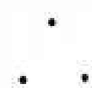
$K_{1,6}$ is not a regular graph.

10.7 Null Graph

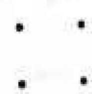
A graph with n vertices and no edge is known as null graph. It is denoted by N_n.

Example :

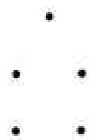
(a) 2 Vertices i.e. N_2



(b) 3 Vertices i.e. N_3



(c) 4 Vertices i.e. N_4



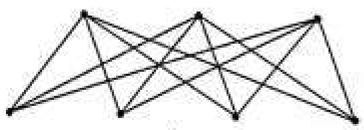
(d) 5 Vertices i.e. N_5

Fig. 10.27

Example 10.9 *Draw a complete bipartite graph, which is not regular.*

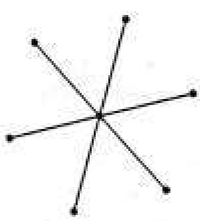
Solution : A complete bipartite graph $K_{m,n}$ is not regular if $m \neq n$.

e.g. $K_{2,3}$, $K_{3,2}$, $K_{3,4}$, $K_{4,3}$, $K_{1,n}$ etc.



$K_{3,4}$

(a)



$K_{1,6}$

(b)

Fig. 10.28

Example 10.10 *Draw a graph which is regular but not bipartite.*

Solution : Complete graph K_4 is regular but not bipartite.

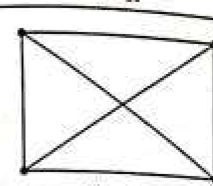


Fig. 10.29

Example 10.11 *How many edges has each of the following graphs*

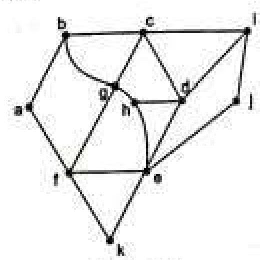
i) K_6 *ii)* $K_{4,6}$

Solution : i) K_6 will have $\frac{6 \times 5}{2} = 15$ edges. As in K_6 all the 6 vertices are of degree 5.

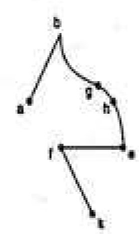
ii) $K_{4,6}$ is a complete bipartite graph hence number of edges $= 4 \times 6 = 24$.

10.8 Subgraphs

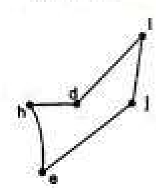
Let G = (V, E) be a graph. A graph H = (V ′, E′) is said to be a subgraph of G if E′ is a subset of E and V ′ is a subset of V. Such that the edges in E′ are incident only with the vertices in V.



(a) Graph G



(b) Sub graph H_1



(c) Sub graph H_2

(d) Sub graph H_3

Fig. 10.30

10.8.1 Properties of Subgraph

1. Every graph is a subgraph of itself.
2. Null graph obtained by deleting all edges of G is also a subgraph of G.
3. A single vertex of graph is also a subgraph of G.
4. A single edge along with end vertices is also a subgraph of G.
5. A subgraph of a subgraph of a graph G is a subgraph of G (i.e. subgraph relation is a transitive relation).

10.8.2 Spanning Subgraph

A subgraph of G is said to be a spanning subgraph if it contains all the vertices of G.

Example :

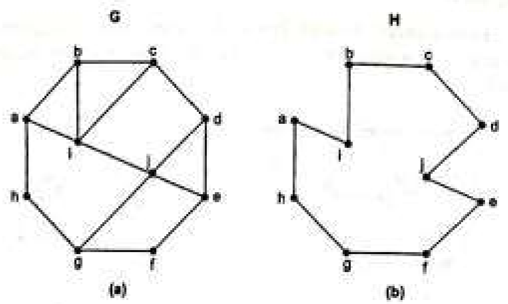


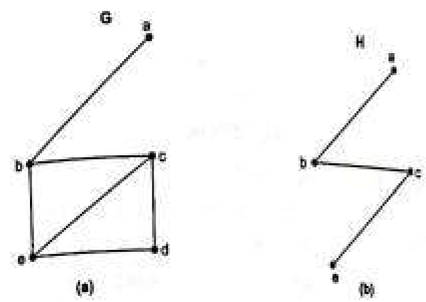
Fig. 10.31

H is a spanning subgraph of G as H contains all the vertices of G.

10.8.3 Complement of Subgraph

The complement of a subgraph $H_1 = (V', E')$ with respect to the graph G is another subgraph, $H_2 = (V'', E'')$ such that $E'' = E - E'$ and contains only those vertices which has end points of E''.

Example :



Complement of H

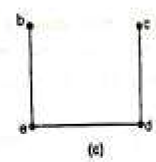


Fig. 10.32

Example 10.12 *Define null subgraph. For the given graph determine whether $H = H(V', E')$ is in a subgraph of G, where*

i) $V' = \{A, B, F\}$ and $E' = \{\{A, B\}, \{A, F\}\}$

ii) $V' = \{B, C, D\}$ and $E' = \{\{B, C\}, \{B, D\}\}$

Solution :

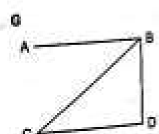
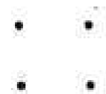


Fig. 10.33

10.8.4 Null Subgraph

A subgraph constructed from a given graph G by deleting all edges of G is also a subgraph defined as null subgraph of G. Null subgraph of G is in 10.33 (a).

i.e. N_4



(a) N_4

Fig. 10.33 (a)

i) $V' = \{A, B, F\}$, $E' = \{\{A, B\}, \{A, F\}\}$

$H = (V', E')$ is not a subgraph of G as F is not a vertex of G.

ii) $V' = \{B, C, D\}$, $E' = \{\{B, C\}, \{B, D\}\}$

$H = (V', E')$ is a subgraph of G as $V' \subseteq V$ and $E' \subseteq E$.

10.9 Factors of a Graph

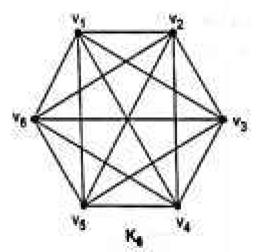
A K-factor of a graph is defined to be a spanning subgraph of the graph with the degree of each of its vertices being K, i.e. a K-factor is regular of degree K.

If G is the sum of K-factors, their union is called a K-factorization and G itself is k-factorable.

When G has a 1-factor, say G_1, it is clear that number of vertices are even and edges of G are point disjoint.

In particular K_{2n+1} can not have a 1-factor but K_{2n} can have 1-factor of graph G.

Example :



K_6
(a)

1-Factor graph

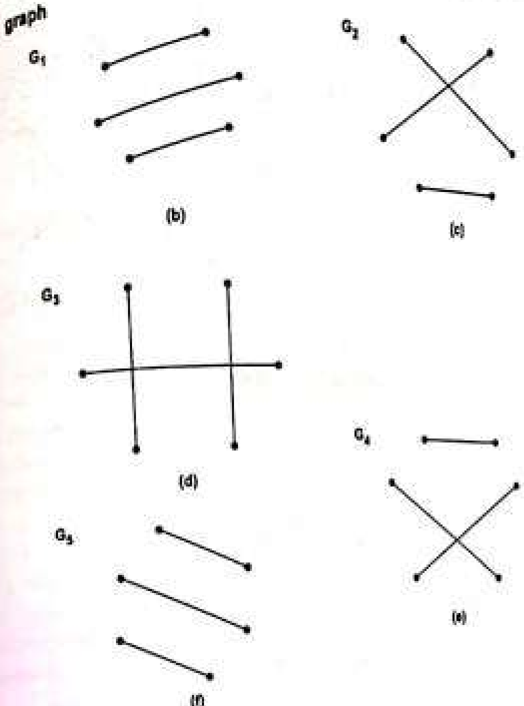
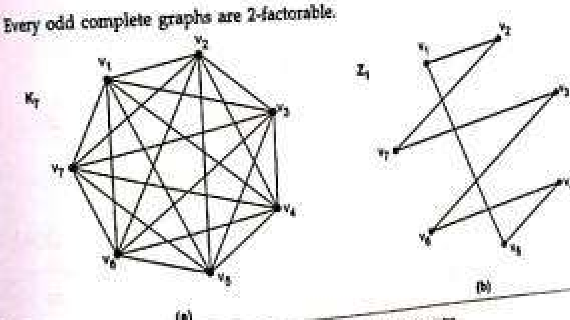
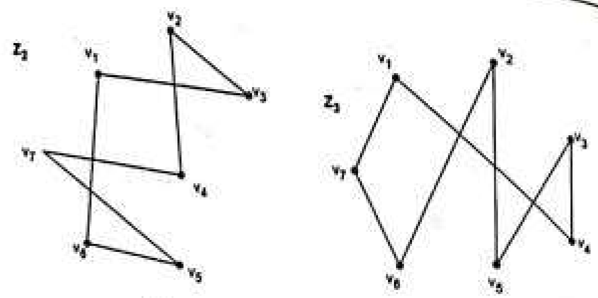


Fig. 10.34

Every odd complete graphs are 2-factorable.





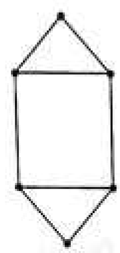
(c) (d)

Fig. 10.35

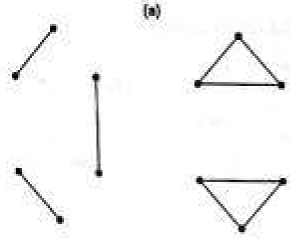
2-Factorization of K_7

It is a spanning circuit.

Example 1 :



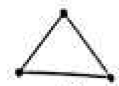
(a)



1 – Factor graph (b) 2 – Factor graph (c)

Fig. 10.36

Example 2 :



K_3

Fig. 10.37

1-factor graph is not possible for K_3 as it is a complete graph with odd number of vertices.

10.10 Isomorphism

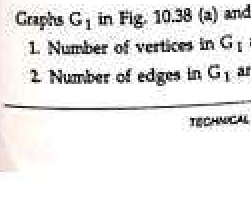
Two graphs G_1 and G_2 are said to be isomorphic if there is a one to one correspondence between their vertices and between their edges such that incidences are preserved.

In other words two graphs are isomorphic if

1. Number of vertices are same.
2. Number of edges are same.
3. If G_1 has n vertices of degree K then G_2 must have exactly n vertices of degree K.
4. Adjacency is preserved.

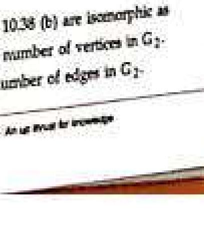
If conditions above are satisfied, then the graphs G_1 and G_2 are known as isomorphic graphs.

Example :



G_1

Fig. 10.38 (a)



G_2

Fig. 10.38 (b)

Graphs G_1 in Fig. 10.38 (a) and G_2 in Fig. 10.38 (b) are isomorphic as

1. Number of vertices in G_1 are same as number of vertices in G_2.
2. Number of edges in G_1 are same as number of edges in G_2.

3. degree of all vertices in G_1 are 3 and degree of all vertices in G_2 are also 3.

Hence G_1 is isomorphic to G_2 and is denoted by

$$G_1 \cong G_2$$

Remark : For the above example G_1 and G_2 are complete graphs K_4 and are also regular graphs.

Remark :

Isomorphism is an equivalence relation on graphs as

1. Every graph is isomorphic to itself.
2. If graph G_1 is isomorphic to G_2 then graph G_2 is isomorphic to G_1.
3. If graph G_1 is isomorphic to G_2 and graph G_2 is isomorphic to graph G_3 then graph G_1 is isomorphic to G_3.

Hence reflexive, symmetric and transitive properties are satisfied, thus the relation isomorphism is an equivalence relation of graphs.

Graphs

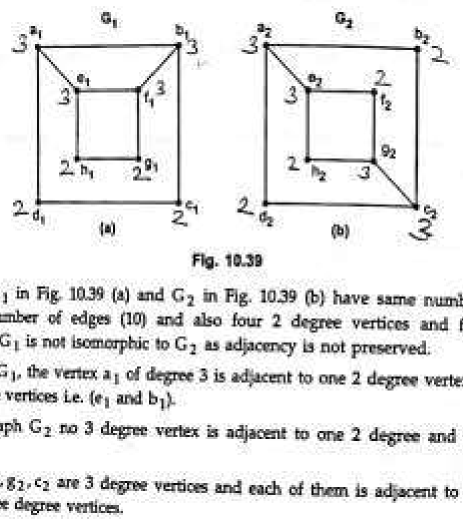


Fig. 10.39

Graphs G_1 in Fig. 10.39 (a) and G_2 in Fig. 10.39 (b) have same number of vertices (8), same number of edges (10) and also four 2 degree vertices and four 3 degree vertices, but G_1 is not isomorphic to G_2 as adjacency is not preserved.

In graph G_1, the vertex a_1 of degree 3 is adjacent to one 2 degree vertex (i.e. d_1) and two 3 degree vertices i.e. (e_1 and b_1).

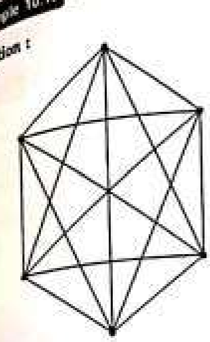
But in graph G_2 no 3 degree vertex is adjacent to one 2 degree and two 3 degree vertices.

As a_2, e_2, g_2, c_2 are 3 degree vertices and each of them is adjacent to two 2 degree and one, three degree vertices.

Hence adjacency is not preserved and thus $G_1 \ncong G_2$.

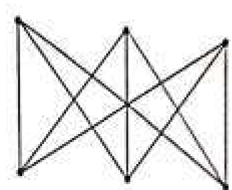
Example 10.13 *Find whether K_6 and $K_{3,3}$ graphs are isomorphic or not.*

Solution :



K_6

Fig. 10.40 (a)



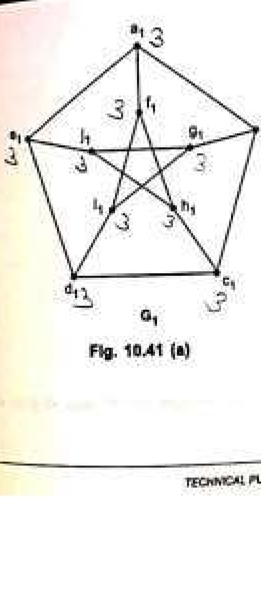
$K_{3,3}$

Fig. 10.40 (b)

K_6 and $K_{3,3}$ are not isomorphic as degree of each vertex of K_6 is 5 while degree of each vertex of $K_{3,3}$ is 3. Hence degree is not preserved, hence

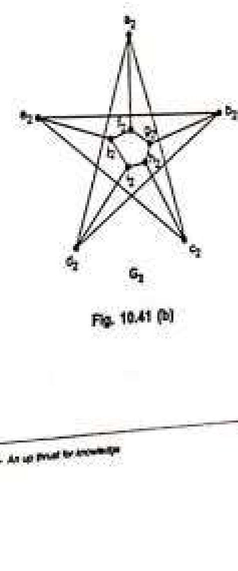
$$K_6 \ncong K_{3,3}$$

Example 10.14 *Show that the following graphs are isomorphic.*



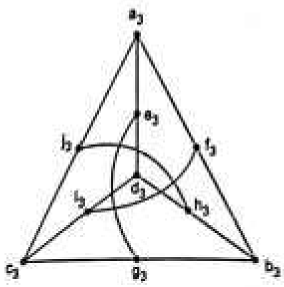
G_1

Fig. 10.41 (a)



G_2

Fig. 10.41 (b)



G_3

Fig. 10.41 (c)

Solution : For graphs G_1 and G_2.

G_1 is a graph with 10 number of vertices and 15 number of edges.

Also G_2 is a graph with 10 number of vertices and 15 number of edges.

In G_1 degree of each vertex is 3 and in G_2 also degree of each vertex is 3.

Also adjacency is preserved and hence,

$G_1 \cong G_2$

$a_1 \rightarrow f_2$

$b_1 \rightarrow g_2$

$c_1 \rightarrow h_2$

$d_1 \rightarrow i_2$

$e_1 \rightarrow j_2$

$f_1 \rightarrow a_2$

$g_1 \rightarrow b_2$

$h_1 \rightarrow c_2$

$i_1 \rightarrow d_2$

$j_1 \rightarrow e_2$

Hence, $G_1 \cong G_2$

Now in G_3 also number of edges are 15, number of vertices are 10 and degree of each vertex is 3.

Also adjacency is preserved and hence

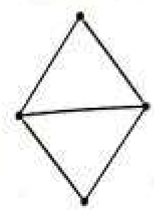
$G_1 \cong G_3$ and $G_2 \cong G_3$

10.11 Complement of a Graph

Let G be a simple graph. The complement of G denoted by $\overline{G}$ is the graph whose vertex set is the same as the vertex set of G and in which two vertices are adjacent if and only if they are not adjacent in G.

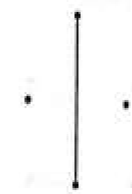
Examples :

1.



G

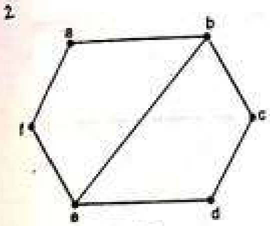
Fig. 10.42 (a)



$\overline{G}$

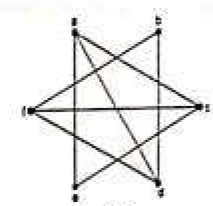
Fig. 10.42 (b)

2.



G

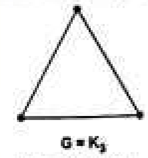
Fig. 10.43 (a)



$\overline{G}$

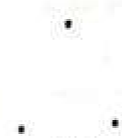
Fig. 10.43 (b)

Complement of a complete graph is a null graph.



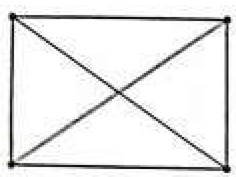
$G = K_3$

Fig. 10.44 (a)



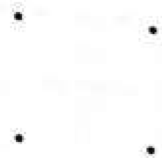
$\overline{G} = N_3$

Fig. 10.44 (b)



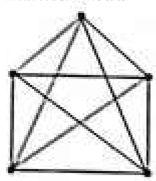
$G = K_4$

Fig. 10.45 (a)



$\overline{G} = N_4$ (Null graph with 4 vertices)

Fig. 10.45 (b)



$G = K_5$

Fig. 10.46 (a)

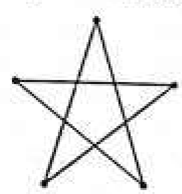
$\overline{G} = N_5$

Fig. 10.46 (b)

A graph is known as **self complementary** if it is isomorphic to its complement.

G

Fig. 10.47 (a)



$\overline{G}$

Fig. 10.47 (b)

$$G \cong G'$$

Hence the graph G is self complementary.

10.12 Non Isomorphic Simple Graphs

With 1 vertex and 0 edge is

Fig. 10.48

With 2 vertex

0 edge

(a)

1 edge

(b)

Fig. 10.49

With 3 vertex

0 edge

(a)

1 edge

(b)

2 edge

(c)

3 edges

(d)

Fig. 10.50

With 4 vertices

0 edge

(a)

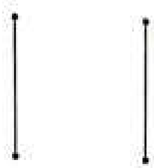
1 edge

(b)

2 edges

(c)

2 edges



(d)

3 edges

(e)

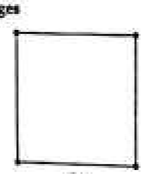
3 edges

(f)

3 edges

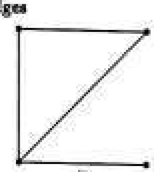
(g)

4 edges



(h)

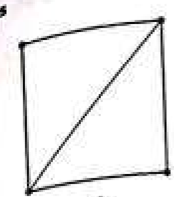
4 edges



(i)

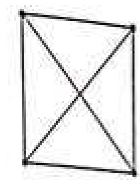
Fig. 10.51

5 edges



(j)

6 edges

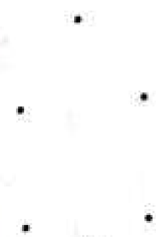


(k)

Fig. 10.51

With 5 vertices

0 edge



(a)

1 edge

(b)

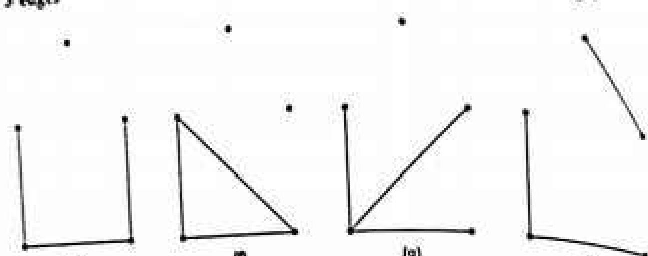
2 edges

(c)

2 edges

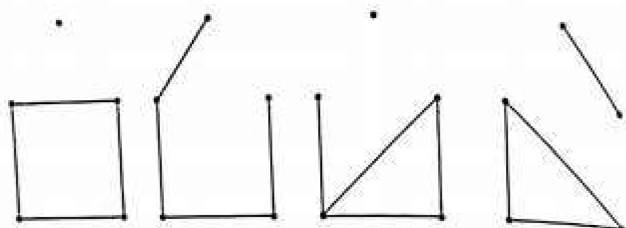
(d)

3 edges 3 edges 3 edges 3 edges

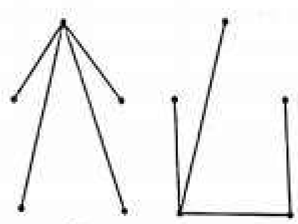


(e) (f) (g) (h)

4 edges

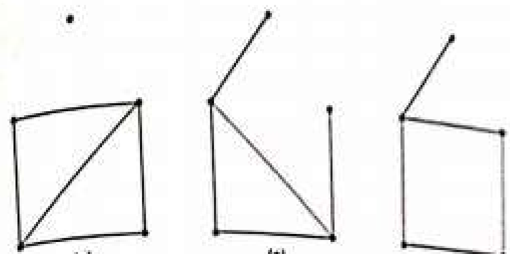


(i) (j) (k) (l)

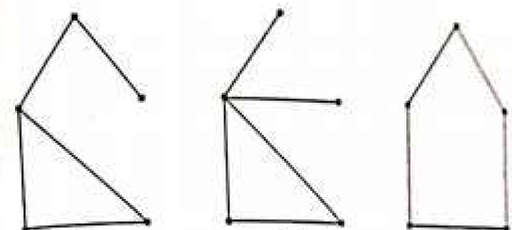


(m) (n)

5 edges



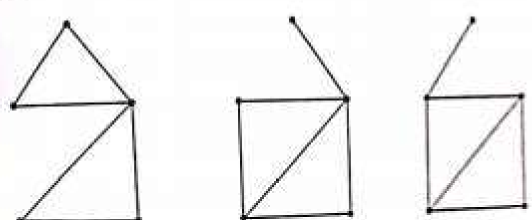
(o) (p) (q)



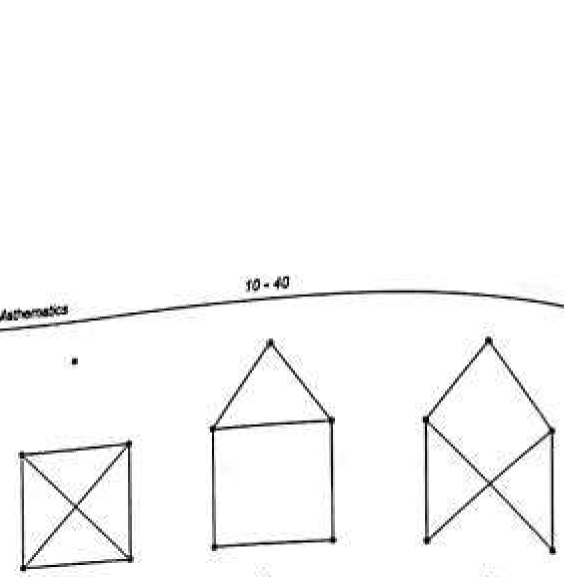
(r) (s) (t)

Fig. 10.52

6 edges

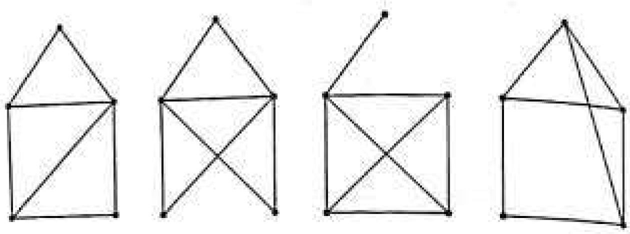


(a) (b) (c)



(d) (e) (f)

7 edges



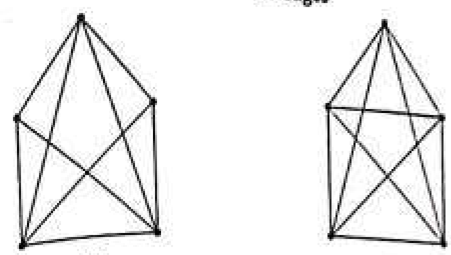
(g) (h) (i) (j)

8 edges

(k) (l)

9 edges

10 edges

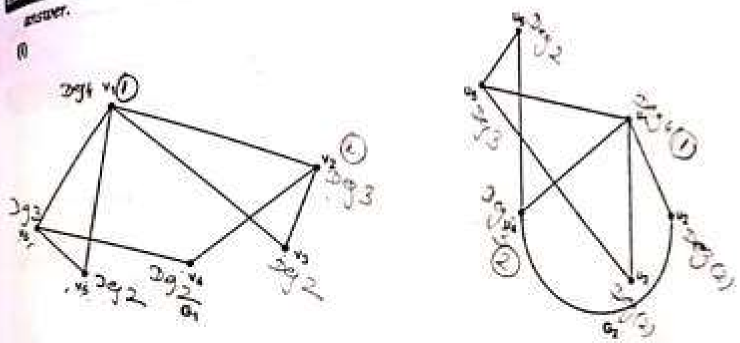


(m) (n)

Fig. 10.53

Example 10.15 *Are the graphs shown in the following figure isomorphic ? Justify your answer.*

(I)



(a) (b)

Fig. 10.54

Solution : Yes, the graphs G_1 and G_2 are isomorphic as

i) G_1 and G_2 both have 6 number of vertices.

ii) Both have 8 number of edges.

iii) G_1 and G_2 have 3 two degree vertices, 2 three degree vertices and 1 four degree vertex.

iv) In G_1 4 degree vertex v_1 is adjacent to 2, three degree and 2, 2 degree vertices. And in G_2 4 degree vertex u_1 is adjacent to 2, three degree and 2, 2 degree vertices hence $v_1 \rightarrow u_1$.

Also in G_1 2 three degree vertices v_2 and v_6 are adjacent to 2, 2 degree vertices and 1, 4 degree vertex, and in G_2 2 three degree vertices u_4 and u_5 are adjacent to 2, 2 degree vertices and 1, 4 degree vertex.

Hence $v_2 \rightarrow u_4$, $v_6 \rightarrow u_3$

Again in G_1 3 two degree vertices are v_3, v_4 and v_5, in which v_5 and v_3 are adjacent to 1, 4 degree vertex and 1, 3 degree vertex. While v_4 is adjacent to 2, three degree vertices and in G_2 3 two degree vertices are u_2, u_3 and u_6 in which u_2 and u_3 are adjacent to 1, 4 degree vertex and 1, 3 degree vertex while u_6 in adjacent to 2, three degree vertices.

Hence, $v_5 \rightarrow u_2$, $v_3 \rightarrow u_3$ and $v_4 \rightarrow u_6$

Thus adjacency is preserved by vertices of G_1 and G_2.

Therefore G_1 and G_2 are isomorphic. $G_1 \cong G_2$.

(ii)

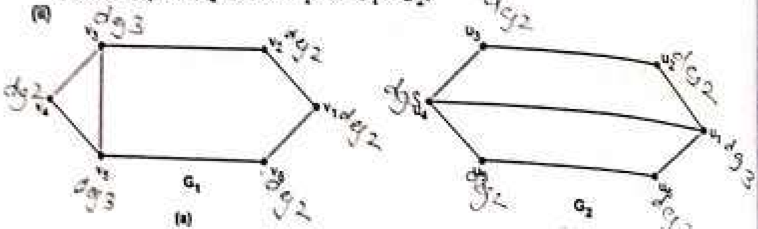


Fig. 10.55

Solution : G_1 is not isomorphic to G_2 as in G_1 the vertex v_1 is adjacent to 2, 2 degree vertices and the vertex v_4 is adjacent two 2, 3 degree vertices but in G_2 no two degree vertex is adjacent to 2, 2 degree or 2, three degree vertices, hence adjacency is not preserved by vertices of G_1 and G_2. Hence G_1 is not isomorphic to G_2.

$G_1 \not\cong G_2$

(iii)

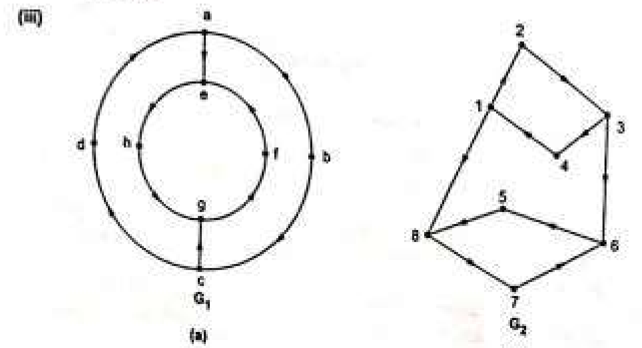


Fig. 10.56

Solution : Both the graphs G_1 and G_2 have same number of vertices 8, same number of edges 10.

Again indegree of h, f, b and d is 1, and outdegree of h, f, b and d is 1.

Indegree of a and c is one and outdegree of a and c is 2.

Indegree of e and g is 2 and outdegree of e and g is 1.

Similarly in G_2, indegree of 2, 4, 5 and 7 is 1 and outdegree of 2, 4, 5, 7 is 1.

Indegree of 1 and 3 is 1 and outdegree of 1 and 3 is 2.

Also indegree of 8 and 6 is 2 and outdegree of 8 and 6 is 1.

Again adjacency is preserved by vertices of G_1 and G_2.

$h \rightarrow 4$, $f \rightarrow 5$, $b \rightarrow 7$, $d \rightarrow 2$

$a \rightarrow 1$, $c \rightarrow 3$, $e \rightarrow 8$, $g \rightarrow 6$

Hence G_1 is isomorphic to G_2.

Therefore, $G_1 \cong G_2$

Example 10.16 *Which of the graphs shown below are isomorphic.*

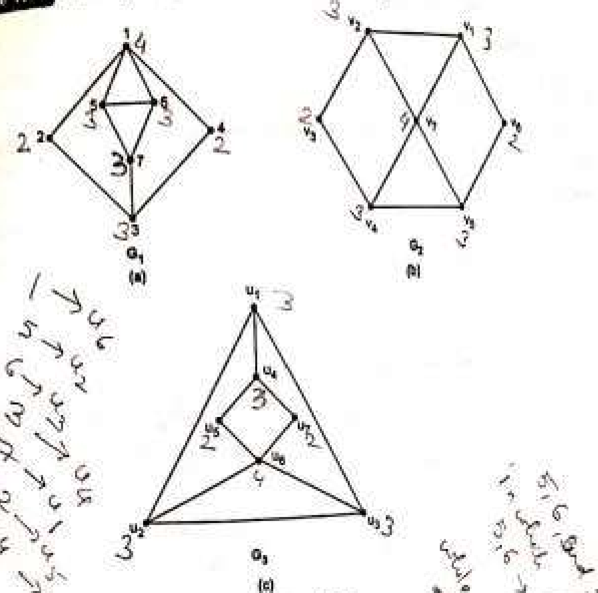


Fig. 10.57

Solution : G_1, G_2 and G_3 graphs are with vertices 7 and edges 10.

All the graphs have 1, 4 degree vertex, 4, 3 degree vertices and 2, 2 degree vertices.

But G_1 is not isomorphic to G_2, also G_3 is not isomorphic to G_2 as in G_2 4 degree vertex is adjacent to 4, 3 degree vertices but in G_1 and G_3, 4 degree vertex is adjacent to 2 three degree and 2, 2 degree vertices, hence adjacency is not preserved.

Therefore,

$$G_1 \ncong G_2$$

also $$G_3 \ncong G_2$$

In case of G_1 and G_3, adjacency is preserved by vertices of G_1 and G_3. As

$$2 \to u_5,\ 4 \to u_7,\ 1 \to u_6,\ 3 \to u_4\ 5 \to u_2,\ 6 \to u_3,\ 7 \to u_1$$

Therefore, $G_1 \cong G_3$

Example 10.17 *Find whether K_4 and $K_{2,2}$ are isomorphic.*

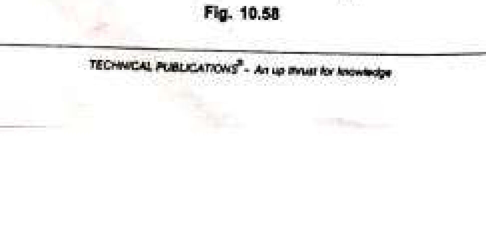
Solution : No, K_4 is not isomorphic to $K_{2,2}$ as number of edges in K_4 are $\frac{4 \times 3}{2}$ i.e. 6 and number of edges in $K_{2,2}$ are $2 \times 2 = 4$.

10.13 Operations on Graphs

If $G = (V, E)$ and $G^* = (V^*, E^*)$ are any two graphs then union, intersection, ring sum etc. operations can be defined on graphs.

where

$$V = \{a_1, b_1, c_1, d_1\} \qquad V^* = \{a_1, b_1, c_1, f_1\}$$
$$E = \{e_1, e_2, e_3, e_4, e_5\} \qquad E^* = \{e_2, e_3, e_6, e_7, e_8, e_9\}$$



(a) (b)

Fig. 10.58

10.13.1 Union of Graphs

For the graphs G and G^*.

Union $G \cup G^*$ is a graph whose vertex set is $V \cup V^*$ and edge set is $E \cup E^*$

where $G = (V, E)$ and $G^* = (V^*, E^*)$

For the Fig. 10.58,

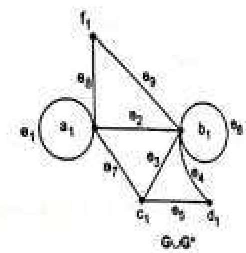


Fig. 10.58 (c)

$G \cup G^*$ is a graph with vertex set $\{a_1, b_1, c_1, d_1, f_1\}$

and edge set $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

10.13.2 Intersection of Graphs

For the graphs G and G^* intersection $G \cap G^*$ is a graph whose vertex set is $V \cap V^*$ and edge set is $E \cap E^*$.

For the graphs G and G^* in Fig. 10.58.

$G \cap G^*$ is a graph with vertex set $\{a_1, b_1, c_1\}$ and edge set $\{e_2, e_3\}$

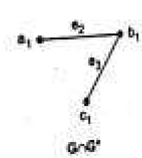


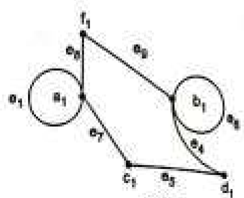
Fig. 10.58 (d)

10.13.3 Ring Sum of Graphs

For the graphs G and G* ring sum G⊕G* is a graph whose vertex set is V∪V* and edge set is set of those edges which are either in G or in G* but not in both.

For the graphs G and G* in Fig. 10.58,

G ⊕ G* is a graph with vertex set $\{a_1, b_1, c_1, d_1, f_1\}$ and edge set $\{e_1, e_4, e_5, e_6, e_7, e_8, e_9\}$



G⊕G*

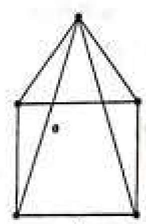
Fig. 10.58 (e)

10.13.4 Removal of an Edge

For a graph G = (V, E).

Let e ∈ E , then the graph G − e is a graph obtained by removing an edge e from the graph.

Example :



G

Fig. 10.59

G − e is the graph

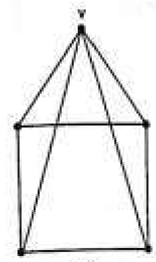
G − e

Fig. 10.59 (a)

10.13.5 Removal of a ...

For a graph G = (V, E).

Let v ∈ V, then the graph G − v is obtained by removing the vertex v from the graph G. Removal of v means, removal of all these edges also which are incident on V.



G

Fig. 10.59 (b)

G − v is the graph.

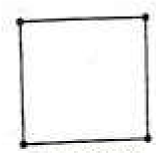


Fig. 10.59 (c)

Example 10.18 *Find the union, intersection and ring sum of the following graphs.*

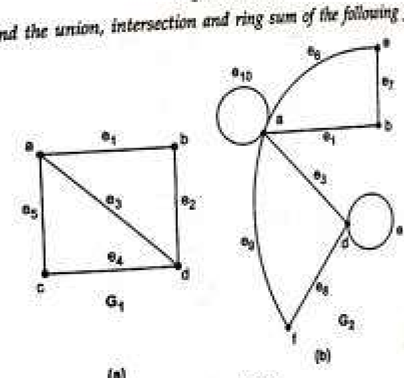
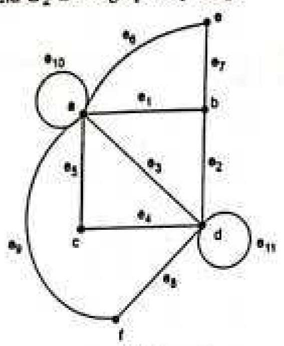


Fig. 10.60

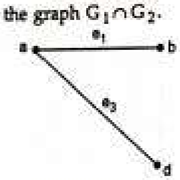
Solution : Union of G_1 and G_2 is the graph $G_1 \cup G_2$.



$G_1 \cup G_2$

Fig. 10.60 (c)

Intersection of G_1 and G_2 is the graph $G_1 \cap G_2$.

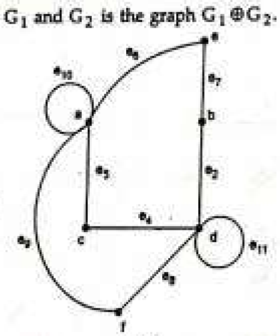


(A∪B)−(A∩B)

$G_1 \cap G_2$

Fig. 10.60 (d)

Ring sum of the graph G_1 and G_2 is the graph $G_1 \oplus G_2$.



$G_1 \oplus G_2$

Fig. 10.60 (e)

Example 10.19 *Find G – v and G – e from the following graph.*

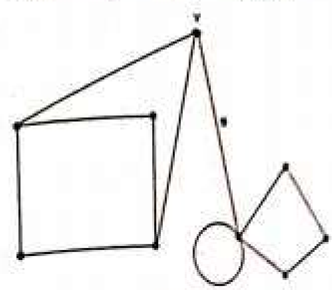


Fig. 10.61

Solution : G – v is a graph.

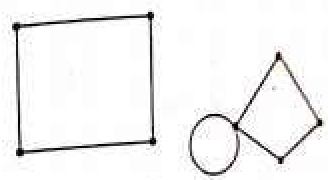


Fig. 10.62

G – e is a graph.

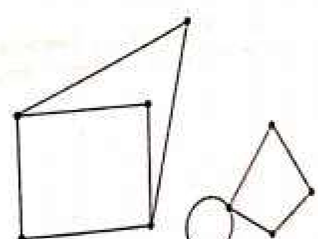


Fig. 10.63

10.14 Path (or Walk) and Circuits

A walk of a graph G is an alternating sequence of vertices and edges $v_1, e_1, v_1, e_2, \ldots v_{n-1}, e_n, v_n$ begining and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it.

This walk (or path) joins v_0 and v_n.

v_0 and v_n are called end vertices and $v_1, v_2, \ldots, v_{n-1}$ are known as interior vertices.

It is some times called as $v_0 - v_n$ walk. If $v_0 = v_n$ the path (or walk) is known as closed path and it is known as open path if $v_0 \neq v_n$. Closed path is also known as circuit.

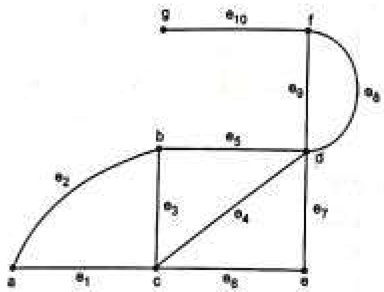


Fig. 10.64

Path I : $a\, e_1\, c\, e_6\, e\, e_7\, d\, e_8\, f\, e_{10}\, g$

Path II : $a\, e_2\, b\, e_5\, d\, e_9\, f\, e_{10}\, g$

Path III : $a\, e_2\, b\, e_3\, c\, e_4\, d\, e_9\, f\, e_{10}\, g$

There are so many other paths from a to g.

These paths are known as open paths as $a \neq g$.

In a simple graph (i.e. a graph without loops and parallel edges), a path may be described by giving only the sequence of vertices traversed in the path.

Example 1 :

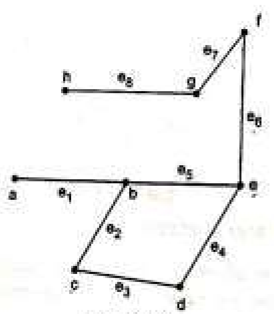


Fig. 10.65

Path I : From (a to h) for Fig. 10.65.

$a\, e_1\, b\, e_2\, c\, e_3\, d\, e_4\, e\, e_6\, f\, e_7\, g\, e_8\, h$

can be written as - abcdefgh

Path II : From a to h for Fig. 10.65

$a\, e_1\, b\, e_5\, e\, e_6\, f\, e_7\, g\, e_8\, h$, can be written as abefgh.

10.14.1 Simple Path

A path in a graph G is called **simple path** if the edges do not repeat in the path.

Path I, II defined in example 1, for Fig. 10.65 are simple paths.

10.14.2 Elementary Path

A path is said to be an **elementary path** if vertices do not repeat in the path.

Path I, II defined in example 1 for Fig. 10.65 are elementary paths.

Fig. 10.66

Path I

$a\, e_1\, b\, e_2\, a\, e_4\, c\, e_3\, a$ in the Fig. 6.66 is not elementary path as vertices repeat in the path.

This is a simple path.

Path II

$a\, e_1\, b\, e_2\, a\, e_4\, c\, e_3\, a\, e_5\, d\, e_6\, a\, e_5\, d\, e_7\, e$ in the Fig. 10.66 is not elementary path as vertices repeat in the path.

Also it is not simple path as edge e_5 repeat in the path.

10.14.3 Circuit

A closed path is known as circuit i.e. $v_0\, e_1\, v_1\, e_2 \ldots\ldots v_{n-1}\, e_n\, v_n$ is circuit if $v_0 = v_n$.

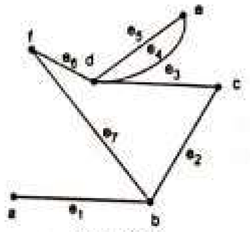


Fig. 10.67

$b\,e_2\,c\,e_3\,d\,e_6\,f\,e_7\,b$ in the Fig. 10.67 is a circuit as end vertices are same.

Also $d\,e_4\,e\,e_5\,d$ in the Fig. 10.67 is a circuit.

10.14.4 Simple Circuits

A circuit in a graph G is called as simple circuit if it does not include the same edge twice.

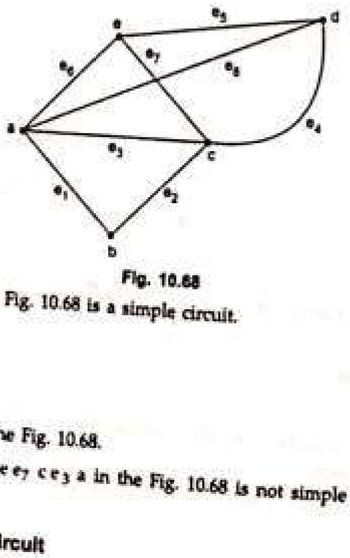


Fig. 10.68

$a\,e_1\,b\,e_2\,c\,e_3\,a$ in the Fig. 10.68 is a simple circuit.

$a\,e_1\,b\,e_2\,c\,e_4\,d\,e_8\,a$

$a\,e_8\,d\,e_5\,e\,e_7\,c\,e_3\,a$

$a\,e_6\,e\,e_7\,c\,e_3\,a$

are simple circuits in the Fig. 10.68.

but $a\,e_6\,e\,e_7\,c\,e_4\,d\,e_5\,e\,e_7\,c\,e_3\,a$ in the Fig. 10.68 is not simple circuit as e_7 edge is included twice.

10.14.5 Elementary Circuit

A circuit is said to be elementary circuit if it does not include the same vertex twice except the end vertex.

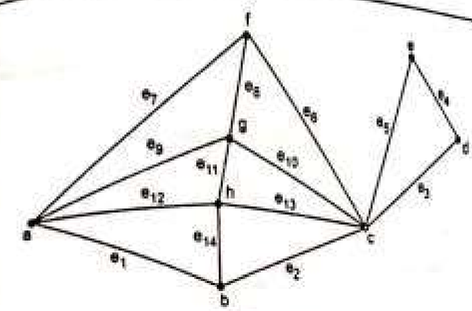


Fig. 10.69

$a\,e_1\,b\,e_2\,c\,e_{13}\,h\,e_{12}\,a$ in the Fig. 10.69 is an elementary circuit as only end vertex a is included twice.

Also $a\,e_7\,f\,e_8\,g\,e_{10}\,c\,e_2\,b\,e_1\,a$. And $a\,e_{12}\,h\,e_{11}\,g\,e_{10}\,c\,e_6\,f\,e_7\,a$ in the Fig. 10.69 are elementary circuits.

But $a\,e_1\,b\,e_2\,c\,e_3\,d\,e_4\,e\,e_5\,c\,e_{13}\,h\,e_{12}\,a$ in the Fig. 10.69 is not elementary circuit as vertex c is included twice.

The length of a path $v_0, v_1, \ldots v_n$ is n, the number of edges in it.

The girth of a graph G, denoted by g (G), is the length of a shortest circuit (if any) in G. The circumference C(G) the length of any longest circuit.

10.15 Reachability

A vertex v of a simple diagraph is said to be reachable from the vertex u of the same diagraph, if there exists a path from u to v.

Geodesic

If vertex v is reachable from the vertex u, then a path of minimum length from u to v is called geodesic.

Distance

The length of a geodesic from vertex u to the vertex v is known as distance and is denoted d (u, v).

The distance from a vertex u to a vertex v, if v is reachable from u is d (u, v) and satisfies following properties .

$$d(u, v) \geq 0 \quad \ldots(1)$$

$$d(u, u) = 0 \quad \ldots(2)$$

Also, $d(u, v) + d(v, w) \geq d(u, w)$ ... (3)

Inequality (3) is known as **triangle inequality** if v is not reachable from u then $d(u, v) = \infty$.

10.16 Connected and Disconnected Graphs

A graph is connected if every pair of vertices are joined by a path otherwise the graph is **disconnected.**

A directed graph is said to be connected if the undirected graph derived from it by ignoring the directions of the edges is connected and is said to be disconnected otherwise.

A disconnected graph consists of two or more components each of which is a connected graph.

A directed graph is said to be **strongly connected** if for every two vertices a and b in the graph there is a path from a to b as well as a path from b to a.

A directed graph (or digraph) is weakly connected if it is not strongly connected and its underlying graph is connected.

A directed graph which is neither strongly connected nor weakly connected is known as disconnected digraph.

Example :

The graph in Fig. 10.70 is a strongly connected graph as there exist a path between every pair of vertices.

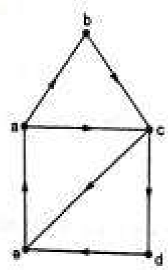


Fig. 10.70

The graph in Fig. 10.71 is not strongly connected as there is no path from e to b.

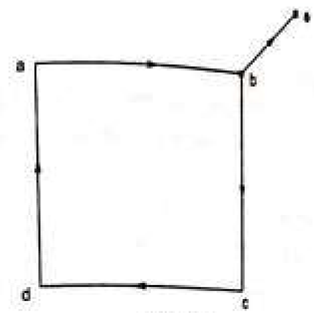


Fig. 10.71

A simple diagraph is known as unilaterally connected if for any pair of vertics of the graph at least one of the vertics of the pair is reachable from the other vertex. If for any pair of vertices of the graph both the vertices of the pair are reachable from one another, then the graph is called strongly connected.

10.16.1 Shortest Path in Weighted Graph

Let (V, E) = G is a weighted graph and w is the weight of an edge, usually referred as the length of the edge.

Consider V as a set of cities and E as a set of highways connecting these cities. It is required to find shortest path between two cities.

In 1959, the algorithm was found by E.W. Dijkstra and is known as Dijkstra's shortest path algorithm.

This algorithm gives shortest path between two vertices.

10.16.2 Dijkstra's Shortest Path Algorithm

Let G = (V, E) be a simple weighted graph. The weight of edge 'e' is refered as length of edge. Let a and z be any two vertices of the graph.

Step 1 : Let T be the set of all vertices, i.e. $T = V$. Let $P = \emptyset$, where P is the set of those vertices which have permanent labels.

In our procedure, we find a shortest path from 'a' to some other vertex, then a shortest path from that vertex to some other vertex till we reach vertex z.

Let $L(a) = 0$, $L(x) = \infty$, $\forall x \in T$ and $x \neq a$.

L denote label.

Step 2 : Select the vertex v in T which has the smallest label.

This label is called the permanent label of v.

Now new $P = P \cup \{v\}$ and $T = T - \{v\}$.

This way continue the process in step 2 till $v = z$.

If $v = z$, then L (z) is the shortest path from the vertex a to z and stop.

Step 3 : If $v \neq z$, then revise the labels of the vertices of T, i.e. the vertices which do not have permanent labels.

The new label of a vertex x in T is given by

$$L(x) = \min \{\text{old } L(x), L(v) + w(v, x)\}$$

where w (v, x) is the weight of the edge joining the vertex v and x.

If there is no direct edge joining v and x then

$$w(v, x) = \infty$$

Step 4 : Repeat steps 2 and 3 until z gets the permanent label.

Example 10.20 *Using Dijkstra's algorithm, find the shortest path between vertices a and z.*

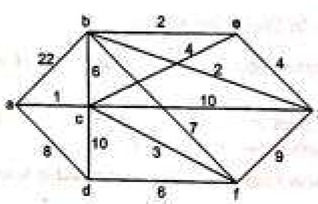


Fig. 10.72

Solution :

1. $P = \emptyset$, $T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0,$

$L\{x\} = \infty, \quad \forall x \in T, \quad x \neq a$

2. $v = a$, the permanent label of a is 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min \{\text{old } L(b), L(a) + w(a, b)\}$

where w (a, b) is the weight of the edge joining the vertex a and b.

If there is no direct edge joining (a, e) then $w(a, e) = \infty$.

$L\{b\} = \min \{\infty, 0 + 22\} = 22$

$L\{c\} = \min \{\infty, 0 + 1\} = 1$

$L\{d\} = \min \{\infty, 0 + 8\} = 8$

$L\{e\} = \min \{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min \{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min \{\infty, 0 + \infty\} = \infty$

3. $v = c$, the permanent label of c is 1.

$P = \{a, c\}, \quad T = \{b, d, e, f, z\}$

$L\{b\} = \min \{\text{old } L(b), L(c) + w(c, b)\}$

$= \min \{22, 1 + 6\} = 7$

$L\{d\} = \min \{8, 1 + 10\} = 8$

$L\{e\} = \min \{\infty, 1 + 4\} = 5$

$L\{f\} = \min \{\infty, 1 + 3\} = 4$

$L\{z\} = \min \{\infty, 1 + 10\} = 11$

4. $v = f$, the permanent label of f is 4.

$P = \{a, c, f\}, \quad T = \{b, d, e, z\}$

$L\{b\} = \min \{7, 4 + 7\} = 7$

$L\{d\} = \min \{8, 4 + 6\} = 8$

$L\{e\} = \min \{5, 4 + \infty\} = 5$

$L\{z\} = \min \{11, 4 + 9\} = 11$

5. $v = e$, the permanent label of e is 5

$P = \{a, c, f, e\}, \quad T = \{b, d, z\}$

$L\{b\} = \min \{7, 5 + 2\} = 7$

$L\{d\} = \min \{8, 5 + \infty\} = 8$

$L\{z\} = \min \{11, 5 + 4\} = 9$

6. $v = b$, the permanent label of b is 7

$P = \{a, c, f, e, b\}, \; T = \{d, z\}$

$L\{d\} = \min \{8, 7 + \infty\} = 8$

$L\{z\} = \min\{9, 7 + \infty\} = 9$

7. v = d the permanent label of d is 8

$P = \{a, c, f, e, b, d\}, T = \{z\}$

$L\{z\} = \min\{9, 8 + \infty\} = 9$

8. v = z, the permanent label of z is 9.

Hence the length of shortest path from a to z is 9.

The shortest path is a c e z.

Fig. 10.73

Example 10.21 *Use Dijkstra's algorithm to find the shortest path between a and z.*

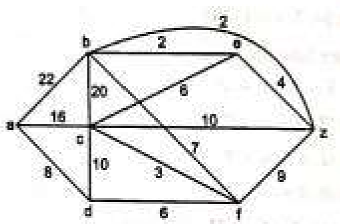


Fig. 10.74

Solution :

1. $P = \varnothing, \quad T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \quad \forall\, x \in T, \quad x \neq a$

2. v = a, the permanent label of a is 0

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 22\} = 22$

$L\{c\} = \min\{\infty, 0 + 16\} = 16$

$L\{d\} = \min\{\infty, 0 + 8\} = 8$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

3. v = d, the permanent label of d is 8.

$P = \{a, d\}, \quad T = \{b, c, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(d) + w(d, b)\}$

$= \min\{22, 8 + \infty\} = 22$

$L\{c\} = \min\{16, 8 + 10\} = 16$

$L\{e\} = \min\{\infty, 8 + \infty\} = \infty$

$L\{f\} = \min\{\infty, 8 + 6\} = 14$

$L\{z\} = \min\{\infty, 8 + \infty\} = \infty$

4. v = f, the permanent label of f is 14.

$P = \{a, d, f\}, \quad T = \{b, c, e, z\}$

$L\{b\} = \min\{\text{old } L(b), L(f) + w(b, f)\}$

$= \min\{22, 14 + 7\} = 21$

$L\{c\} = \min\{16, 14 + 3\} = 16$

$L\{e\} = \min\{\infty, 14 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 14 + 9\} = 23$

5. v = c, the permant lable of c is 16.

$P = \{a, d, f, c\}, \quad T = \{b, e, z\}$

$L(b) = \min\{\text{old } L(b), L(f) + w(f, b)\}$

$= \min\{21, 16 + 20\} = 21$

$L(e) = \min\{\infty, 16 + 6\} = 22$

$L(z) = \min\{23, 16 + 10\} = 23$

6. v = b, the permanent label of b is 21.

P = {a, d, f, c, b}, T = {e, z}

$$L(e) = \min\{\text{old } L(e),\ L(b) + w(e, b)\}$$
$$= \min\{22, 21 + 2\} = 22$$
$$L(z) = \min\{23, 21 + 2\} = 23$$

7. v = e, the permanent label of e is 22.

P = {a, d, f, c, b, e}, T = {z}

$$L(z) = \min\{\text{old } L(z),\ L(e) + w(e, z)\}$$
$$= \min\{23, 22 + 4\} = 23$$

8. v = z, the permanent label of z is 23.

Hence the length of the shortest path from a to z is 23.

The shortest path is adfz or adfbz.

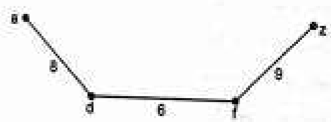


Fig. 10.75 (a)

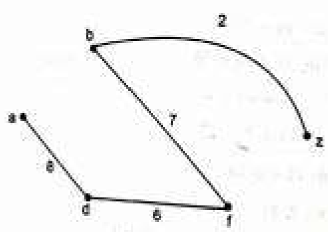


Fig. 10.75 (b)

Example 10.22 *Find the shortest path from a-z in the given graph using Dijkstra's algorithm.*

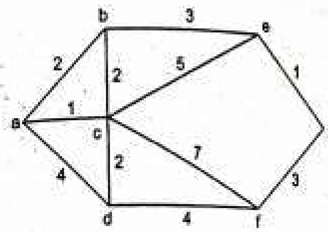


Fig. 10.76

Solution :

1. $P = \varnothing$, T = {a, b, c, d, e, f, z}

$$L(a) = 0$$
$$L(x) = \infty,\ \forall x \in T,\ x \neq a$$

2. v = a, the permanent label of a is 0.

P = {a}, T = {b, c, d, e, f, z}

$$L(b) = \min\{\text{old } L(b),\ L(a) + w(a, b)\}$$
$$= \min\{\infty, 0 + 2\} = 2$$
$$L(c) = \min\{\infty, 0 + 1\} = 1$$
$$L(d) = \min\{\infty, 0 + 4\} = 4$$
$$L(e) = \min\{\infty, 0 + \infty\} = \infty$$
$$L(f) = \min\{\infty, 0 + \infty\} = \infty$$
$$L(z) = \min\{\infty, 0 + \infty\} = \infty$$

3. v = c, the permanent label of c is 1.

P = {a, c}, T = {b, d, e, f, z}

$$L(b) = \min\{2, 1 + 2\} = 2$$
$$L(d) = \min\{4, 1 + 2\} = 3$$
$$L(e) = \min\{\infty, 1 + 5\} = 6$$
$$L(f) = \min\{\infty, 1 + 7\} = 8$$
$$L(z) = \min\{\infty, 1 + \infty\} = \infty$$

4. v = b, the permanent label of b is 2.

P = {a, c, b} , T = {d, e, f, z}

$$L\{d\} = \min\{3, 2 + \infty\} = 3$$
$$L\{e\} = \min\{6, 2 + 3\} = 5$$
$$L\{f\} = \min\{8, 2 + \infty\} = 8$$
$$L\{z\} = \min\{\infty, 2 + \infty\} = \infty$$

5. v = d, the permanent label of d is 3.

P = {a, c, b, d}, T = {e, f, z}

$$L\{e\} = \min\{5, 3 + \infty\} = 5$$
$$L\{f\} = \min\{8, 3 + 4\} = 7$$
$$L\{z\} = \min\{\infty, 3 + \infty\} = \infty$$

6. v = e, the permanent label of e is 5.

P = {a, c, b, d, e}, T = {f, z}

$$L\{f\} = \min\{7, 5 + \infty\} = 7$$
$$L\{z\} = \min\{\infty, 5 + 1\} = 6$$

7. v = z, the permant label of z is 6.

Hence the length of shortest path from a to z is 6.

The shortest path is a b e z.

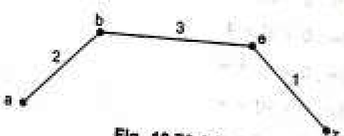


Fig. 10.76 (a)

Example 10.23 *Find the shortest path between a-z for the given graph : using Dijkstra's algorithm.*

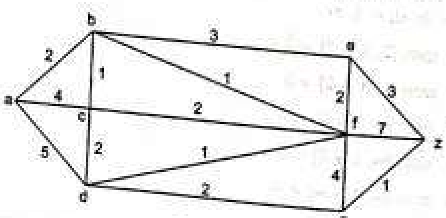


Fig. 10.77

Solution :

1. P = Ø , T = {a, b, c, d, e, f, g, z}

$$L\{a\} = 0$$
$$L\{x\} = \infty, \quad \forall\, x \in T, \quad x \neq a.$$

2. v = a, the permanent label of a is 0.

P = {a}, T = {b, c, d, e, f, g, z}

$$L\{b\} = \min\{\text{old } L(b),\ L(a) + w(a, b)\}$$
$$= \min\{\infty, 0 + 2\} = 2$$
$$L\{c\} = \min\{\infty, 0 + 4\} = 4$$
$$L\{d\} = \min\{\infty, 0 + 5\} = 5$$
$$L\{e\} = L\{f\} = L\{g\} = L\{z\} = \infty$$

3. v = b

P = {a, b}, T = {c, d, e, f, g, z}

$$L\{c\} = \min\{L(c), L(b) + w\{b, c\}\}$$
$$= \min\{4, 2 + 1\} = 3$$
$$L\{d\} = \min\{5, 2 + \infty\} = 5$$
$$L\{e\} = \min\{\infty, 2 + 3\} = 5$$
$$L\{f\} = \min\{\infty, 2 + 1\} = 3$$
$$L\{g\} = L\{z\} = \infty$$

4. v = c or f

Let v = f, permanent label of f = 3.

P = {a, b, f}, T = {c, d, e, g, z}

$$L\{c\} = \min\{3, 3 + 2\} = 3$$
$$L\{d\} = \min\{5, 3 + 1\} = 4$$
$$L\{e\} = \min\{5, 3 + 2\} = 5$$
$$L\{g\} = \min\{\infty, 3 + 4\} = 7$$
$$L\{z\} = \min\{\infty, 3 + 7\} = 10$$

5. $v = c$, permanent label of $c = 3$.

$P = \{a, b, f, c\}$, $T = \{d, e, g, z\}$

$$L\{d\} = \min\{4, 3 + 2\} = 4$$
$$L\{e\} = \min\{5, 3 + \infty\} = 5$$
$$L\{g\} = \min\{7, 3 + \infty\} = 7$$
$$L\{z\} = \min\{10, 3 + \infty\} = 10$$

6. $v = d$, permanent label of $d = 4$.

$P = \{a, b, f, c, d\}$, $T = \{e, g, z\}$

$$L\{e\} = \min\{5, 4 + \infty\} = 5$$
$$L\{g\} = \min\{7, 4 + 2\} = 6$$
$$L\{z\} = \min\{10, 4 + \infty\} = 10$$

7. $v = e$, permanent label of e is 5.

$P = \{a, b, f, c, d, e\}$, $T = \{g, z\}$

$$L\{g\} = \min\{6, 5 + \infty\} = 6$$
$$L\{z\} = \min\{10, 5 + 3\} = 8$$

8. $v = g$, permanent label of g is 6.

$P = \{a, b, f, c, d, e, g\}$, $T = \{z\}$

$$L\{z\} = \min\{8, 6 + 1\} = 7$$

9. $v = z$, permanent label of z is 7.

Hence the length of shortest path from a to z is 7.

The shortest path is a b f d g z.

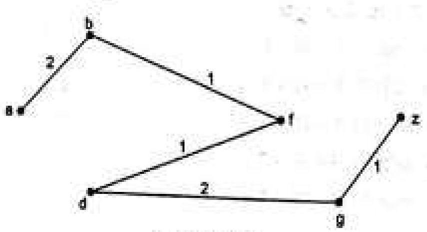


Fig. 10.77 (a)

Example 10.24 *State the Dijkstra's algorithm to obtain the shortest path (distance) between two vertices in the given graph and apply the same to obtain the shortest path between a and z in the following graph.*

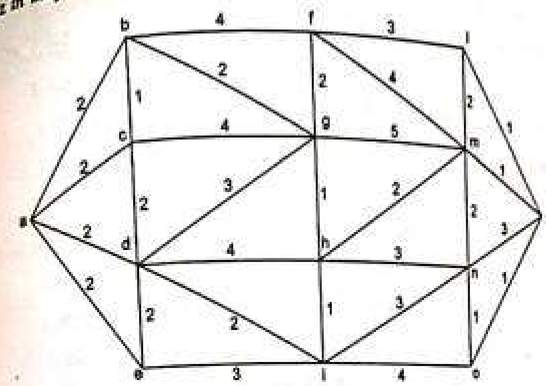


Fig. 10.78

Solution : Refer section 10.16.2 for Dijkstra's algorithm

1. $P = \emptyset$, $T = \{a, b, c, d, e, f, g, h, i, l, m, n, o, z\}$

$$L(a) = 0$$
$$L(x) = \infty, \quad \forall\, x \in T, \quad x \neq a.$$

2. $v = a$, the permanent label of a is o

$P = \{a\}$, $T = \{b, c, d, e, f, g, h, i, l, m, n, o, z\}$

$$L(b) = \min\{\text{old } L(b), L(a) + w(a, b)\}$$
$$L(b) = \min\{\infty, 0 + 2\} = 2$$
$$L\{c\} = \min\{\infty, 0 + 2\} = 2$$
$$L\{d\} = \min\{\infty, 0 + 2\} = 2$$
$$L\{e\} = \min\{\infty, 0 + 2\} = 2$$
$$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$$
$$L\{g\} = \min\{\infty, 0 + \infty\} = \infty$$
$$L\{h\} = \min\{\infty, 0 + \infty\} = \infty$$
$$L\{i\} = \min\{\infty, 0 + \infty\} = \infty$$

$L\{l\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{m\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{n\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{o\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$

3. v = b, the permanent label of b is 2.

$P = \{a, b\}, \quad T = \{c, d, e, f, g, h, i, l, m, n, o, z\}$

$L\{c\} = \min\{2, 2 + 1\} = 2$

$L\{d\} = \min\{2, 2 + \infty\} = 2$

$L\{e\} = \min\{2, 2 + \infty\} = 2$

$L\{f\} = \min\{\infty, 2 + 4\} = 6$

$L\{g\} = \min\{\infty, 2 + 2\} = 4$

$L\{h\} = L\{i\} = L\{l\} = L\{m\} = L\{n\} = L\{o\} = L\{z\} = \infty.$

4. v = c, the permanent label of c is 2.

$P = \{a, b, c\}, \quad T = \{d, e, f, g, h, i, l, m, n, o, z\}$

$L\{d\} = \min\{2, 2 + 2\} = 2$

$L\{e\} = \min\{2, 2 + \infty\} = 2$

$L\{f\} = \min\{6, 2 + \infty\} = 6$

$L\{g\} = \min\{4, 2 + 4\} = 4$

$L\{h\} = L\{i\} = L\{l\} = L\{m\} = L\{n\} = L\{o\} = L\{z\} = \infty$

5. v = d, the permanent label of d is 2

$P = \{a, b, c, d\}, \quad T = \{e, f, g, h, i, l, m, n, o, z\}$

$L\{e\} = \min\{2, 2 + 2\} = 2$

$L\{f\} = \min\{6, 2 + \infty\} = 6$

$L\{g\} = \min\{4, 2 + 3\} = 4$

$L\{h\} = \min\{\infty, 2 + 4\} = 6$

$L\{i\} = \min\{\infty, 2 + 2\} = 4$

$L\{l\} = L\{m\} = L\{n\} = L\{o\} = L\{z\} = \infty$

6. v = e, the permanent label of e is 2

$P = \{a, b, c, d, e\}, \quad T = \{f, g, h, i, l, m, n, o, z\}$

$L\{f\} = \min\{6, 2 + \infty\} = 6$

$L\{g\} = \min\{4, 2 + \infty\} = 4$

$L\{h\} = \min\{6, 2 + \infty\} = 6$

$L\{i\} = \min\{4, 2 + 3\} = 4$

$L\{l\} = L\{m\} = L\{n\} = L\{o\} = L\{z\} = \infty$

7. v = g, the permanent label of g is 4.

$P = \{a, b, c, d, e, g\}, \quad T = \{f, h, i, l, m, n, o, z\}$

$L\{f\} = \min\{6, 4 + 2\} = 6$

$L\{h\} = \min\{6, 4 + 1\} = 5$

$L\{i\} = \min\{4, 4 + \infty\} = 4$

$L\{l\} = \infty$

$L\{m\} = \min\{\infty, 4 + 5\} = 9$

$L\{n\} = L\{o\} = L\{z\} = \infty$

8. v = i, the permanent label of i is 4.

$P = \{a, b, c, d, e, g, i\}, \quad T = \{f, h, l, m, n, o, z\}$

$L\{f\} = \min\{6, 4 + \infty\} = 6$

$L\{h\} = \min\{5, 4 + 1\} = 5$

$L\{l\} = \min\{\infty, 5 + \infty\} = \infty$

$L\{m\} = \min\{9, 5 + \infty\} = 9$

$L\{n\} = \min\{\infty, 5 + 3\} = 8$

$L\{o\} = \min\{\infty, 5 + 4\} = 9$

$L\{z\} = \min\{\infty, 5 + \infty\} = \infty$

9. v = h, the permanent label of h is 5.

$P = \{a, b, c, d, e, g, i, h\}, \quad T = \{f, l, m, n, o, z\}$

$L\{f\} = \min\{6, 5 + \infty\} = 6$

$L\{l\} = \min\{\infty, 5 + \infty\} = \infty$

$L\{m\} = \min\{9, 5 + 2\} = 7$

$$L(n) = \min\{8, 5+3\} = 8$$
$$L(o) = \min\{9, 5+\infty\} = 9$$
$$L(z) = \min\{\infty, 5+\infty\} = \infty$$

10. $v = f$, the permanent label of f is 6.

$P = \{a, b, c, d, e, g, i, h, f\}$, $T = \{l, m, n, o, z\}$

$$L(l) = \min\{\infty, 6+3\} = 9$$
$$L(m) = \min\{7, 6+4\} = 7$$
$$L(n) = \min\{8, 5+\infty\} = 8$$
$$L(o) = \min\{9, 5+\infty\} = 9$$
$$L(z) = \min\{\infty, 5+\infty\} = \infty$$

11. $v = m$, the permanent label of m is 7.

$P = \{a, b, c, d, e, g, i, h, f, m\}$, $T = \{l, n, o, z\}$

$$L(l) = \min\{9, 7+2\} = 9$$
$$L(n) = \min\{8, 7+2\} = 8$$
$$L(o) = \min\{9, 7+\infty\} = 9$$
$$L(z) = \min\{\infty, 7+1\} = 8$$

12. $v = z$, the permanent label of z is 8.

Hence the length of shortest path from a to z is 8.

The shortest path is a b g h m z.

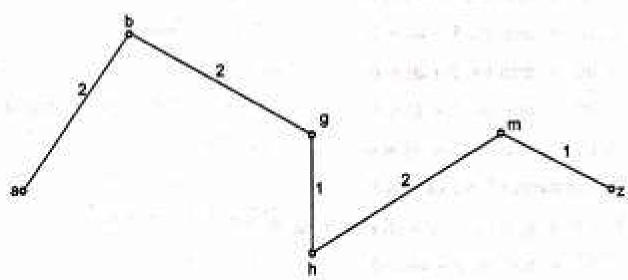


Fig. 10.78 (a)

10.17 Eulerian Path and Eulerian Circuit

Swiss mathematician Leonhard Euler became the father of the theory of graphs when he proved in 1736 that it was not possible to cross each of the seven bridges on the river Pregel in Königsberg, Germany, once and only once in a walking tour.

Two islands C and D formed by Pregel river in Königsberg were connected to each other and to the banks A and B with seven bridges. The problem was to start at any of the four land areas of the city. A, B, C or D, walk over each of the seven bridges exactly once and return to the starting point.

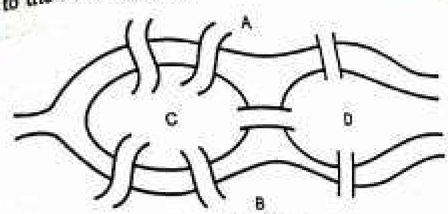


Fig. 10.79

In proving that the problem is unsolvable, Euler replaced each land area by a vertex and each bridge by an edge joining the corresponding vertices.

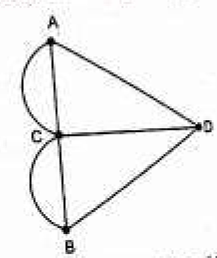


Fig. 10.80 Equivalent graphical representation of Königsberg bridge problem

10.17.1 Eulerian Path

Definition : A path is known as **Eulerian path** if every edge of the graph G appears exactly once in the path.

10.17.2 Eulerian Circuit

Definition : The circuit which contains every edge of the graph exactly once is called **Eulerian circuit.**

10.17.3 Eulerian Graph

A graph which has an Eulerian circuit is known as **Eulerian graph.**

Example 1 :

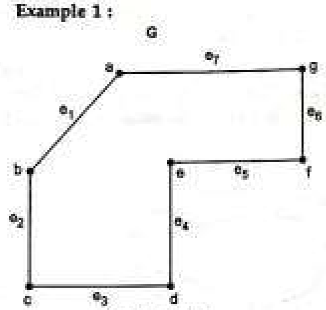


Fig. 10.81

G is an Eulerian graph as G contains Eulerian circuit.

a e_1 b e_2 c e_3 d e_4 e e_5 f e_6 g e_7 a

Example 2 :

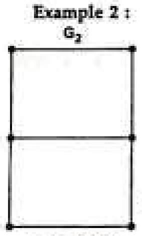


Fig. 10.82

G_2 contains Eulerian path but not Eulerian circuit.

From example 2 it is clear that a graph contains Eulerian path need not have Eulerian circuit.

The existence of Eulerian paths or Eulerian circuits in a graph is related to the degree of vertices.

We show now a result due to Euler.

Theorem 1 : An undirected graph possesses an Eulerian path if and only if it is connected and has either zero or two vertices of odd degree.

Theorem 2 : An undirected graph possesses an Eulerian circuit if and only if it is connected and its vertices are all of even degree.

Theorem 3 : A directed graph possesses an Eulerian circuit if and only if it is connected and the incoming degree of every vertex is equal to its outgoing degree.

Theorem 4 : A directed graph possesses an Eulerian path if and only if it is connected and the incoming degree of every vertex is equal to its outgoing degree with the possible exceptation of two vertices. For these two vertices, the incoming degree of one is one more then its outgoing degree and the incoming degree of the other is one less than its outgoing degree.

Remark :

In Königsberg bridge all the 4 vertices A, B, C and D are of odd degree hence from theorem 2 there does not exist any Eulerian circuit.

Hence it is not possible to walk through all seven bridges exactly once and come back to original position.

Example 10.25 *Draw a graph which contains an Eulerian circuit.*

Solution :

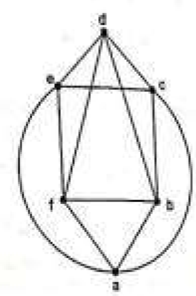


Fig. 10.83

The above graph in Fig. 10.83 has an Eulerian circuit because the graph is connected and all the vertices are of even degree.

Example 10.26 *Draw a graph which contains an Eulerian path but does not contain an Eulerian circuit.*

Solution :

The graph in Fig. 10.84 is connected and there are exactly two vertices b and f of odd degree, hence the graph in Fig. 10.84 has an Eulerian path but not Eulerian circuit.

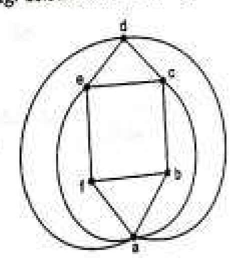


Fig. 10.84

Example 10.27 *For what values of n does K_n, the complete graph on n nodes, have an Euler circuit ? For which it have an Euler path ?*

Solution : In K_n, each vertex is joined to remaining (n – 1) vertices, i.e. the degree of each vertex is (n – 1). If n is odd then the degree of each vertex will be even. Hence, by theorem 1 on Euler circuit, the complete graph K_n will contain an Euler circuit.

For an Euler path, the graph should have either zero or exactly 2 vertices of odd degree.

Exactly two vertices of odd degree is possible only in K_2.

K_2

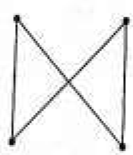
Also, zero vertices of odd degree are possible in K_n when n is odd. Hence K_2 and all complete graphs K_n when n is odd have an Euler path.

Example 10.28 *Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit.*

Solution : In complete bipartite graph $K_{m,n}$ consider the following cases.

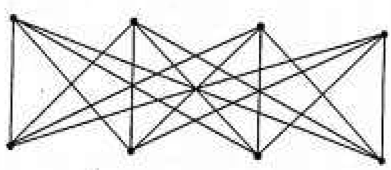
(i) When m = n and both m and n are even, then degree of each vertex is even and hence the graph $K_{m,n}$ will contain an Eulerian circuit.

Examples $K_{2,2}$, $K_{4,4}$



$K_{2,2}$

(a)

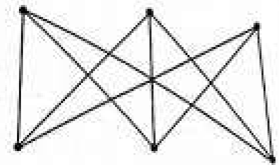


$K_{4,4}$

(b)

Fig. 10.85

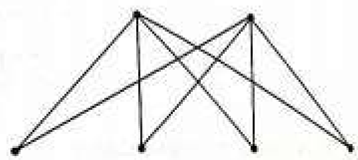
(ii) If m = n and both are odd, then the degree of each vertex is odd. Hence, the graph will not contain an Eulerian circuit.



$K_{3,3}$

Fig. 10.86

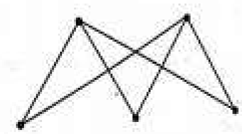
(iii) If m ≠ n and both m and n are even, then the graph has an Eulerian circuit. For instance, $K_{2,4}$ has an Eulerian circuit.



$K_{2,4}$

Fig. 10.87

(iv) If m ≠ n and either m is odd or n is odd or both are odd, then the graph will not possess an Eulerian circuit. $K_{2,3}$, $K_{3,5}$ are the examples of graphs which do not possess an Eulerian circuit.



$K_{2,3}$

Fig. 10.88

Example 10.29 *Which of the following graphs possess Euler's path or circuit ?*

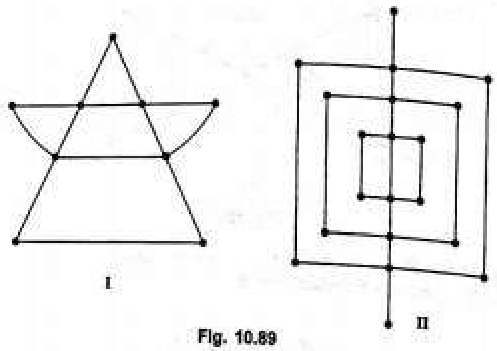


Fig. 10.89

Solution : In a graph I, each vertex is of even degree. Hence it possess an Euler's circuit. Graph II is connected graph and it has exactly 2 vertices of odd degree. Hence this graph possess an Euler's path.

10.18 Hamiltonian Path and Hamiltonian Circuit

A problem similar to the determination of an Eulerian Path or an Eulerian Circuit is to determine a path or a circuit that passes through each vertex in a graph once and only once. We define a Hamiltonian path (circuit) to be a path (circuit) that passes through each of the vertices in a graph exactly once. (Sir William Hamilton invented the game "all around the world" in which the player is asked to determine a route along the dodecahedron that will pass through each angular point once and only once).

10.18.1 Hamiltonian Path

Definition : A path in a connected graph G is a Hamiltonian Path if it contains every vertex of G exactly once.

10.18.2 Hamiltonian Circuit

Definition : A circuit in a connected graph G is a Hamiltonian circuit if it contains every vertex of G exactly once except the first and the last vertex.

10.18.3 All Around the World

A game invented by Sir William Hamilton in 1859 uses a regular solid dodecahedron whose 20 vertices are labelled with the names of famous cities. The player is challenged to travel "around the world" by finding a closed circuit along the edges which passes through each vertex exactly once.

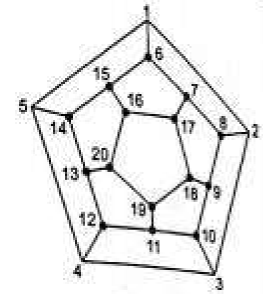


Fig. 10.90

One such Hamiltonian circuit is 2, 3, 10, 11, 12, 4, 5, 1, 6, 7, 17, 16, 15, 14, 13, 20, 19, 18, 9, 8, 2.

Unlike the Eulerian paths and circuits, there are no necessary and sufficient conditions to determine Hamiltonian circuits or paths.

However, there are some results that guarantee the existence of Hamiltonian path or circuit.

Theorem 1 Let G be a simple connected graph on n vertices. If the sum of the degree for each pair of vertices in G is $(n - 1)$ or large, then there exists a Hamiltonian path in G.

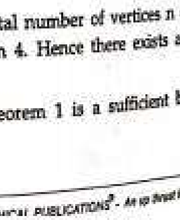
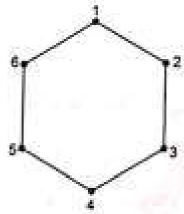


Fig. 10.91

For the above Fig. 10.91, total number of vertices $n = 5$ and degree sum of every pair of vertices is 4 or greater than 4. Hence there exists a Hamiltonian path in G which is given by abcde.

Remark : Condition in theorem 1 is a sufficient but not a necessary condition for existence of path in a graph.



G

Fig. 10.92

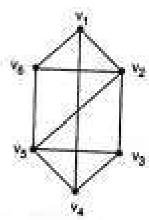
G has a Hamiltonian path 123456 but degree sum of each pair of vertex is $4 \ngeq (6-1) = (n-1)$

Example 10.30 *Consider the problem of scheduling seven examinations in seven days so that two examinations given by the same instructor are not scheduled on consecutive days. If no instructor gives more than four examinations, show that it is always possible to schedule the examinations.*

Solution : Let G be a group with seven vertices corresponding to the seven examinations. There is an edge between any two vertices which correspond to two examinations given by different instructors. Since the degree of each vertex is at least 3, the sum of the degrees of any two vertices is at least 6. Consequently, G always contains a Hamiltonian path, which corresponds to a suitable schedule for the seven examinations.

Theorem 2 : If G = (V, E) is a simple connected graph on vertices and if degree of each vertex v is greater than or equal to $\frac{n}{2}$ then G will contain a Hamiltonian circuit.

For example :



G

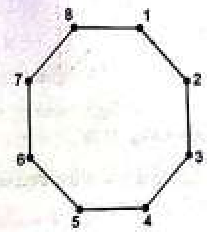
Fig. 10.93

In G the degree of each vertex is greater than or equal to 3.

Hence by theorem 2 on Hamiltonian Circuits, the graph has a Hamiltonian circuit $v_1v_2v_3v_4v_5v_6v_1$

Remark : The condition given in theorem 2 is just a sufficient condition for the given graph to have a Hamiltonian circuit.

For example :



G

Fig. 10.94

G with 8 vertices has a Hamiltonian circuit 1 2 3 4 5 6 7 8 1 but degree of each vertex is 2.

$$\deg(v) = 2 \ngeq \frac{8}{2} = 4.$$

Theorem 3 : Let G = (V, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subset S of V, the components in the graph (G – S) is less than or equal to the number of vertices in S.

For example : A complete bipartite graph $K_{m,n}$ does not have a Hamiltonian circuit when $n \neq m$

When m = n, it has a Hamiltonian circuit.

Example 10.31 *Hamiltonian circuit exists in complete bipartite graph. Justify your answer.*

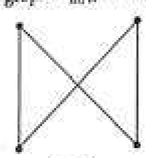
Solution : Hamiltonian circuit exists in complete bipartite graph $K_{m,n}$ for which m = n and Hamiltonian circuit does not exists in complete bipartite graph $K_{m,n}$ for which $m \neq n$.

Consider $K_{m,n}$ $m \neq n$

Let (V_1, V_2) be a partition of the vertex set of $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$ and m < n. The graph $K_{m,n} - V_1$ is a null graph on n vertices and hence it is a disconnected graph with n components.

Therefore, the number of components in $K_{m,n} - V_1 = n \leq |V_1| = m$ which is contradiction to $m < n$. Hence by theorem 3, $k_{m,n}$ does not contain a Hamiltonian circuit when $m \neq n$.

Consider the complete bipartite graph $K_{m,n}$ when $m = n = 2$, i.e. $K_{2,2}$.



$G = K_{2,2}$

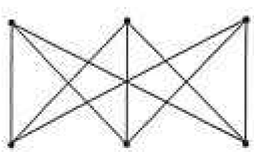
Fig. 10.95

Total number of vertices in graph G is 4 and the degree of each vertex v is 2 which satisfies the condition for theorem 3

$$\deg v = 2 \geq \frac{4}{2}$$

Hence $K_{2,2}$ has a Hamiltonian circuit.

Similarly for $K_{3,3}$



$G = K_{3,3}$

Fig. 10.96

$\deg v = 3 \geq \frac{6}{2}$, where number of vertices are 6.

Hence $K_{3,3}$ has Hamiltonian circuit.

10.19 Travelling Salesman Problem

In this section, we discuss a natural extension of the problem of finding a Hamiltonian circuit in a graph.

The problem is known as "Travelling Salesman Problem", which is stated as below.

A salesperson wants to have a tour of the n cities which starts and ends at the same city and includes visiting each of the remaining (n–1) cities once and only once. Moreover, an itinerary that has a minimum total distance is desired.

The above problem can be represented by a weighted graph, in which vertices represent the cities and the roads between the cities represent the edges.

The weight of each edge is the distance between two cities.

We shall refer to w (i, j) as the length of the edge {i, j}.

The travelling salesperson problem asks for a Hamiltonian circuit of minimum length, where the length of a circuit is defined as the sum of the lengths of the edges in the circuit.

For any three vertices i, j, k

w (i, j) + w (j, k) ≥ w (i, k) [Triangular inequality]

The travelling salesperson problem turns out to be a difficult one if n is very large and also in case of complete graph K_n the problem can be solved theoretically by listing all possible $\frac{(n-1)!}{2}$ Hamiltonian circuits and picking the one which has the least weight.

Here we present a procedure known as the nearest neighbour method which give reasonably good results for the travelling salesperson problem.

10.20 Nearest - Neighbour Method

In this method, we start the Hamiltonian circuit with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we get the Hamiltonian circuit.

The procedure is discribed as -

1. Start with any arbitrary vertex, and find the vertex that is closest to the starting vertex to form an initial path of one edge. Construct this path by selecting different vertices as discribed in step (2).
2. Let x denote the latest vertex that was added to the path. Among all vertices that are not in the path, pick the one that is closest to x, and add to the path the edge connecting x and this vertex. Repeat this step until all vertices in the graph G are included in the path.
3. For m a circuit by adding the edge connecting the starting vertex and the last vertex added.

Example 10.32 *For the following figure solve the following options:*

a) Use nearest neighbourhood method to find out Hamiltonian circuit for the graph in the following Fig. 10.97 starting at vertex a.

b) Repeat the part (a), starting at vertex 'd' instead.

c) Determine the minimum Hamiltonian circuit for the graph in the following Fig. 10.97.

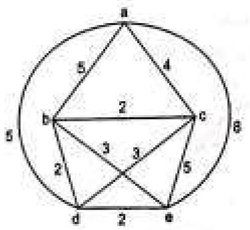


Fig. 10.97

Solution : a) Start with vertex a. There are 4 adjacent vertices b, c, d and e. The closest vertex is c (as length of the edge (a, c) = 4, which is minimum).

(i) Initial path = {a, c}

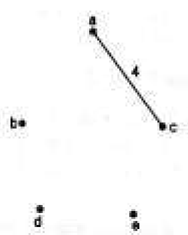


Fig. 10.97 (a)

(ii) There are three vertices adjacent to c, namely b, d, e (except a). The closest is b

Path = {a, c, b}

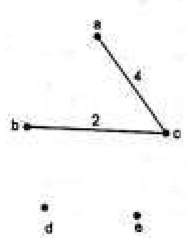


Fig. 10.97 (b)

(iii) There are two vertices adjacent to b, namely d and e. The closest is d

Path = {a, c, b, d}

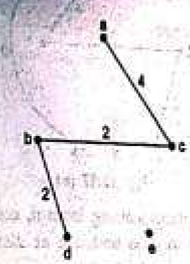


Fig. 10.97 (c)

(iv) path = {a, c, b, d, e}

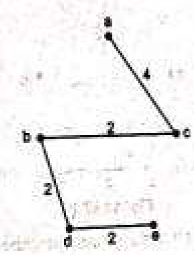


Fig. 10.97 (d)

(v) Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from e to a.

Hamiltonian circuit = {a, c, b, d, e, a}

The weight of Hamiltonian circuit = 18

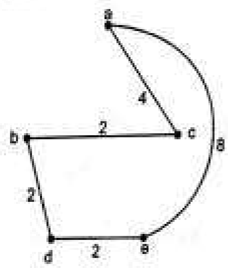


Fig. 10.97 (e)

(b) To find the Hamiltonian circuit starting from d, consider the vertex d. Since both the vertices b and e are nearest to d as both are at distance 2 from d, hence we can choose any vertex.

Let e be the vertex.

(i) Initial path = {d, e}

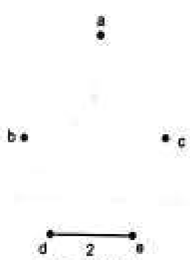


Fig. 10.97 (f)

(ii) a, b, c are adjacent to e but b is the nearest neighbour of e.

Path = {d, e, b}

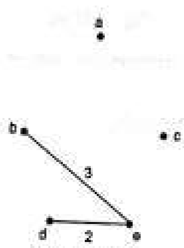


Fig. 10.97 (g)

(iii) Path = {d, e, b, c}

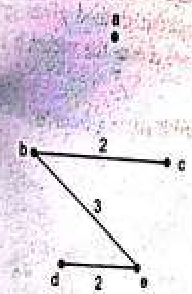


Fig. 10.97 (h)

(iv) Path = {d, e, b, c, a}

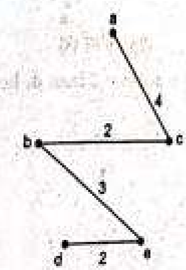


Fig. 10.97 (i)

(v) Since all the vertices are traversed, to complete the Hamiltonian circuit, there should be a path from a to d.

Hamiltonian = {d, e, b, c, a, d}

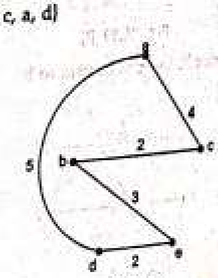


Fig. 10.97 (j)

The weight of the Hamiltonian circuit is 16.

Remark : In fact, no efficient algorithm is available to find the solution of travelling salesperson problem. Nearest neighbour method gives good result but not always best result as in the above problem if we choose the vertex b next to the vertex d, we get the following solution.

b) To find the Hamiltonian circuit starting from 'd'.

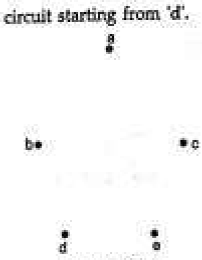


Fig. 10.97 (k)

b and e both vertices are at the distance 2 from d, hence we can choose any vertex. Let the vertex be b.

(i) Initial path = {d, b}

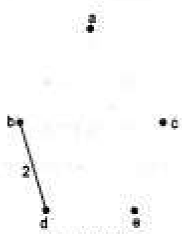


Fig. 10.97 (l)

(ii) a, c, e are adjacent to vertex b, but c is nearest to b.

Path = {d, b, c}

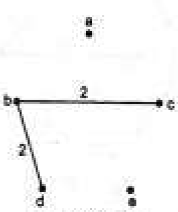


Fig. 10.97 (m)

(iii) a and e are adjacent vertices to c but a is nearest to c.

Path = {d, b, c, a}

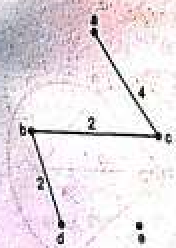


Fig. 10.97 (n)

(iv) Path = {d, b, c, a, e}

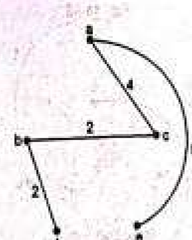


Fig. 10.97 (o)

(v) Since all the vertices are traversed, to complete the Hamiltonian circuit there should be a path from e to d.

Hamiltonian circuit = {d, b, c, a, e, d}

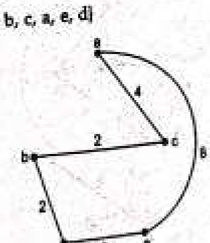


Fig. 10.97 (p)

The weight of the Hamiltonian circuit is 18.

Weight of which is more than the Hamiltonian circuit {d e b c a d}

c) The minimum Hamiltonian circuit is with weight 16.

Example 10.33 *Use nearest neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.*

Solution : (i) Initial path = {a, d}

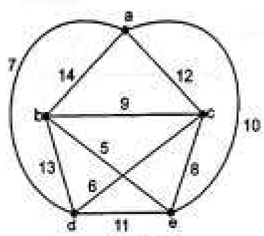


Fig. 10.98

(ii) Path = {a, d, c}

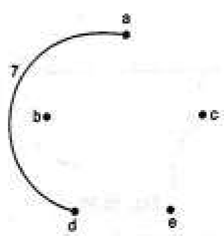


Fig. 10.98 (a)

(iii) Path = {a, d, c, e}

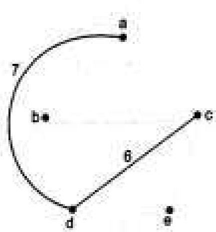


Fig. 10.98 (b)

(iv) = {a, d, c, e, b}

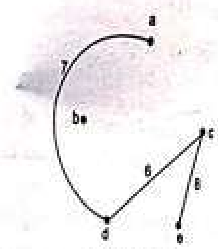


Fig. 10.98 (c)

(v) Hamiltonian circuit = {a, d, c, e, b, a}

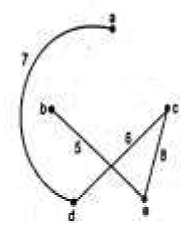


Fig. 10.98 (d)

Weight of the Hamiltonian circuit = 40.

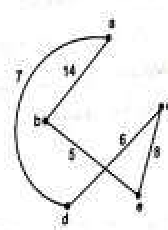


Fig. 10.98 (e)

Example 10.34 *Show that the complete graph K_n ($n \geq 3$) has a Hamiltonian circuit. What is the length of that circuit? How many Hamiltonian circuits exist in K_n?*

Solution : The complete graph K_n ($n \geq 3$) has n vertices and the degree of each vertex is (n – 1)

As $n \geq 3$

$n - 1 \geq \frac{n}{2}$

$\Rightarrow d(v) \geq \frac{n}{2}, \ \forall \ v \in V$ of K_n

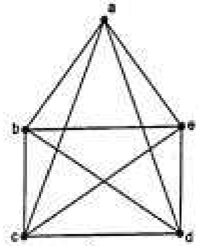
Therefore the theorem 2 on Hamiltonian Circuit is satisfied

Hence the complete graph on n vertices has a Hamiltonian circuit.

For example, Consider K_5

The Hamiltonian circuit is a b c d e a.

The length of the Hamiltonian circuit is the number of vertices in the circuit. For K_5,



K_5

Fig. 10.99

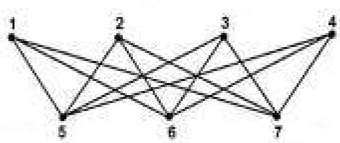
the Hamiltonian circuit will contain 5 vertices.

Hence in K_n, the length of the Hamiltonian circuit is n

There are $\frac{(n-1)!}{2}$ Hamiltonian circuits in the complete graph K_n.

Example 10.35 *Find a Hamiltonian path and a Hamiltonian circuit in* $K_{4,3}$.

Solution : The complete bipartite graph $K_{4,3}$ is given by



$K_{4,3}$

Fig. 10.100

Here the total number of vertices in the graph $K_{4,3}$ is 7. Also the degree sum of any pair of vertices $\geq (7-1)$. Hence by theorem 1 on Hamiltonian path the graph $K_{4,3}$ has a Hamiltonian path. It is given by

1 5 2 6 3 7 4

Also the graph $K_{4,3}$ does not contain a Hamiltonian circuit because $m \neq n$.

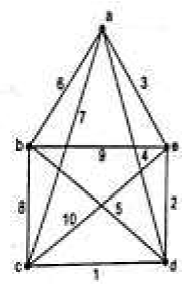
Example 10.36 *(i) Give an example of a graph that has both an Eulerian circuit and Hamiltonian circuit.*

(ii) Give an example of a graph that has an Eulerian circuit but no Hamiltonian circuit.

(iii) Give an example of a graph that has Hamiltonian circuit but no Eulerian circuit.

(iv) Give an example of a graph that has neither Hamiltonian circuit nor Eulerian circuit.

Solution : (i) Graph with Eulerian circuit and Hamiltonian circuit is as shown in Fig. 10.101.



K_5

Fig. 10.101

Eulerian Circuit is 1 2 3 4 5 6 7 8 9 10 1. Hamiltonian circuit is a b c d e a.

Remark : Every complete graph K_n where n is odd is an example which contains both Eulerian circuit and Hamiltonian circuit.

(ii) Graphs with Eulerian circuit but no Hamiltonian circuit are shown in Fig. 10.102 and Fig. 10.103.

a) Eulerian circuit b e a b c d b. No Hamiltonian circuit.

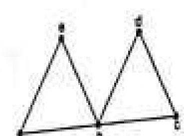


Fig. 10.102

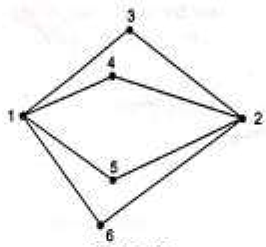


Fig. 10.103

(b) Eulerian circuit 1 6 2 5 1 4 2 3 1. No Hamiltonian circuit.

(iii) Graph with Hamiltonian circuit but no Eulerian circuit is shown in Fig. 10.104

No Eulerian circuit. Hamiltonian circuit is a b c d e f a

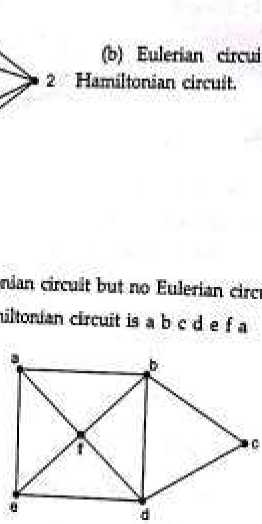


Fig. 10.104

(iv) Graphs with neither Hamiltonian nor Eluerian circuit are shown in Fig. 10.105 and Fig. 10.106.

(i)

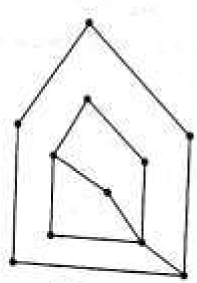
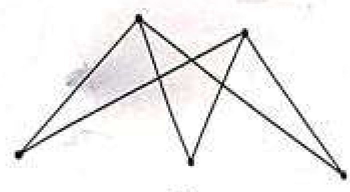


Fig. 10.105

(ii)



$K_{2,3}$

Fig. 10.106

10.21 Edge and Vertex Connectivity

In this section, we will see how removal of edges and vertices disconnects the graph. We will also define the terms edge connectivity and vertex connectivity which are useful in construction of communication network.

10.21.1 Edge Connectivity

In a connected graph G, a cut set is a minimal set of edges whose removal disconnects the graph and increases the components of the graph by one.

In other words, a cut set in a connected graph G is a set of edges whose removal from G leaves G disconnected provided removal of no proper subset of these edges disconnects G.

Example :

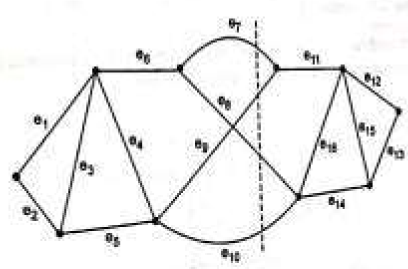


Fig. 10.107

In the Fig. 10.107 $\{e_7, e_8, e_9, e_{10}\}$ is a cut set

Other cut set are $\{e_6, e_9, e_{10}\}$, $\{e_{11}, e_8, e_{10}\}$ etc.

For the cut set $\{e_7, e_8, e_9, e_{10}\}$ the above graph G will have 2 components.

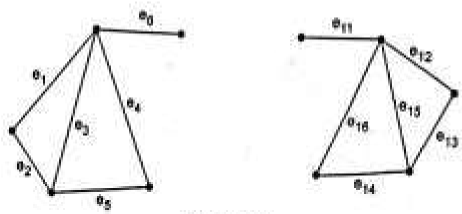


Fig. 10.108

Hence is S is a cut set then (G – S) has exactly two components.

If the cut set of the connected graph contains only one edge, then that edge is called a bridge (or isthums), i.e. the removal of a bridge (or isthums) disconnect the graph.

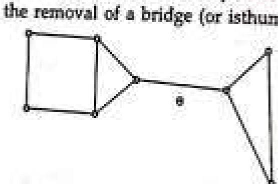


Fig. 10.109

edge e is an isthmus (or bridge) in the graph in Fig. 10.109.

Connectivity : The connectivity $\lambda(G)$ of a graph G is the minimum number of edges whose removal results in a disconnected or trivial (or null) graphs.

10.21.2 Vertex Connectivity

The vertex connectivity K(G) of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.

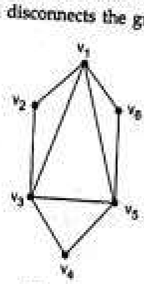


Fig. 10.110

If we remove vertices v_1 and v_3, the graph G will be disconnected.

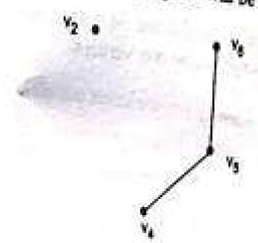


Fig. 10.111

A graph is said to be K- connected if its vertex connectivity is K

A graph is known as **separable graph** if its vertex connectivity is one.

In a separable graph, a vertex whose removal disconnects the graph is said to be a cut vertex or a cut point.

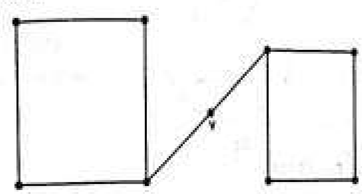


Fig. 10.112

In the above graph Fig. 10.112 v is known as cut vertex.

Remark : The complete graph K_n can not be disconnected by removing any number of vertices, but the null graph results after removing n–1 vertices.

Therefore $K(K_n) = n-1$.

Theorem 1 : For any graph G, $K(G) \leq \lambda(G) \geq \delta(G)$

where $\delta(G)$ is the minimum degree of a vertex in the graph G.

Theorem 2 : If G has n vertices and $\delta(G) \geq \left[\frac{n}{2}\right]$ then $\lambda(G) = \delta(G)$

Theorem 3 : Among all graphs with n vertices and e edges, the maximum connectivity is 0 when $e < n - 1$ and is $\left\lfloor\frac{2e}{n}\right\rfloor$ when $e \geq n-1$.

10.22 Planar Graph

A graph is said to be planar graph if it can be drawn on a plane in such a way that no edges cross one another, except at common vertices.

Examples :

1.

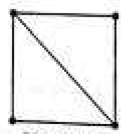
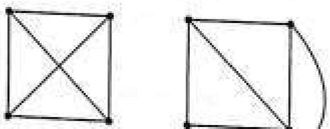


Fig. 10.113

2.



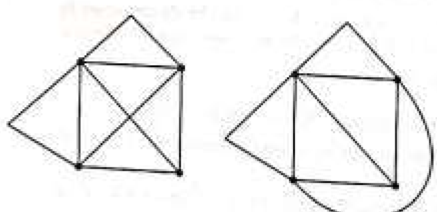
(a) (b)

Fig. 10.114

Graph in 10.114 (a) can be redrawn as 10.114 (b)

Hence Fig. 10.114 (b) is a planar graph

3.



(a) (b)

Fig. 10.115

Fig. 10.115 (b) is planar graph

10.22.1 Regions

A planar representation of a graph divides the plane into regions (also known as windows faces and meshes.)

A region characterized by the set of edges forming its boundary.

The unbounded region is known as exterior region (or exterior face).

Examples :

1.

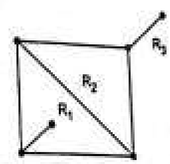


Fig. 10.116

The planar graph of Fig. 10.116 has three regions R_1, R_2 and R_3. R_3 is known as exterior region, R_1 and R_2 are known as bounded regions.

Remark : Bounded region is sometimes known as finite region as the area is finite.

Unbounded region is also known as infinite region as the area is infinite. A planar graph has exactly one infinite region.

2.

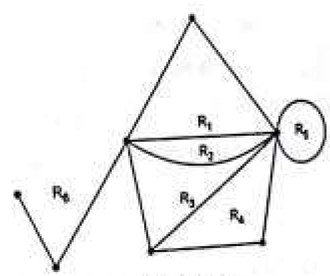


Fig. 10.117

R_1, R_2, R_3, R_4 and R_5 are bounded regions. R_6 is unbounded region.

The subject of planar graph was discovered by Euler in his investigation of polyhedra.

The Euler formula for polyhedra is one of the classical results of mathematics.

Theorem (Euler polyhedra formula)

For any connected planar graph,

$$\boxed{v - e + r = 2}$$

Where v, e, r are the total number of vertices, edges and regions in the graph respectively.

Proof of Euler's theorem

Let G be any connected planar graph with v vertices, e edges and r number of regions.

The proof uses induction on the number of edges.

(i) Basis of Induction : For two graphs with single edge, the result $v - e + r = 2$ is satisfied.

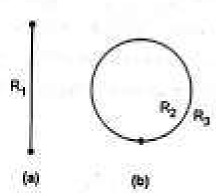


Fig. 10.118

In Fig. 10.118 (a) v = 2, e = 1 and r = 1

$v - e + r = 2.$

For Fig. 10.118 (b) v = 1, e = 1, r = 2

$v - e + r = 2.$

(ii) Induction Step

(i) Assume that $v - e + r = 2$ is satisfied in all graphs with (n − 1) edges.

(ii) Let G be a connected graph with n edges. If G has a vertex of degree 1, the removal if this vertex together with the edge incident with it will yield a connected graph G′ Fig. 10.119 (a).

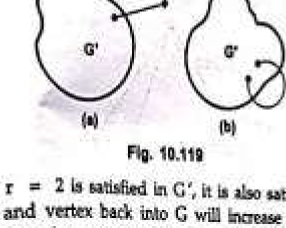


Fig. 10.119

(iii) Since $v - e + r = 2$ is satisfied in G′, it is also satisfied in G, because putting removed edge and vertex back into G will increase count of vertices by 1 and count of edges by 1, but would not alter the count of regions.

(iv) If G has no vertex of degree 1, the-removal of any edge in the boundary of a finite region will yield a connected graph G′ (Fig. 10.119 (b))

(v) Since this equation is satisfied in G′, it will also be satisfied in G because putting the removed edge back into G′ will not alter the count of vertices but will change count of edge by one and count of regions by 1.

Corollary 1 : If G = (V, E) is simple connected planar graph then $e \leq 3v - 6$ where e is total number of edges and v is total number of vertices in the graph G.

Proof : Since the graph is a simple planar graph, therefore each region of the planar graph is bounded by at least three or more edges. Hence the total number of edges $e \geq 3r$. Also, each edge is included in at least 2 regions of the planar graph G therefore $2e \geq 3r$.

or $$\boxed{\frac{2e}{3} \geq r}$$

Substitute this value of r in Euler's formula, we get

$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

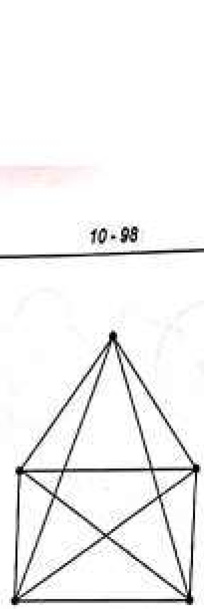
i.e. $$\boxed{e \leq 3v - 6}$$

Using corollary 1, we show that the graph K_5 (complete graph with 5 vertices and 10 edges). ($e = \frac{5 \times 4}{2} = 10$)

$3v - 6 = 15 - 6 = 9$

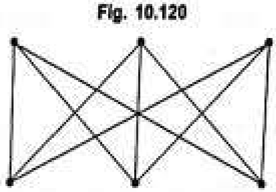
But $9 \not\geq 10$

Hence K_5 is non planar.



K_5

Fig. 10.120



$K_{3,3}$

Fig. 10.121

We shall show that the graph $K_{3,3}$ is not a planar graph either. If the graph $K_{3,3}$ is planar, every region would be bounded by four or more edged. Thus from $v - e + r = 2$ (as $2e \geq 4r$)

$$v - e + \frac{2e}{4} \geq 2$$

or $\quad 2v - 4 \geq e$

For the graph $K_{3,3}$, $v = 6$ and $e = 9$

Then $8 \geq 9$

Which is impossible. Hence $k_{3,3}$ **is not a planar graph.** Thus $K_{3,3}$ **is a non planar graph.**

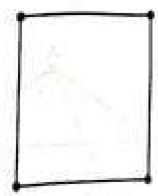
K_5 is known as **Kuratowski's first graph** and $K_{3,3}$ is known as **Kuratowski's second graph.**

Remarks : Although Euler's formula can be used to assert whether a given graph is non planar, still the application of formula can become tricky for graphs containing even a moderate number of vertices and edges.

Moreover, except by actually mapping a graph on the plane, we have seen no asserting that a given graph is planar.

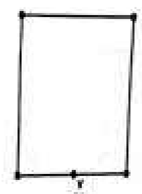
Kuratowski's theorem enables us to determine unequivocally the planarity graph.

1. The planarity of a graph is clearly not affected if an edge is divided into edges by the insertion of a new vertex of degree 2.



G

Fig. 10.122

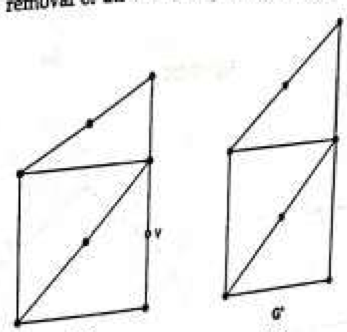


G'

Fig. 10.123

Vertex v is inserted in G to get G'.

2. If two edges that are incident with a vertex of degree 2 are combined as a singl edge by the removal of the vertex, the planarity of a graph is not affected.



G (a) G' (b)

Fig. 10.124

Vertex v of degree 2 is removed from graph G to get graph G'.

Two graphs G and G' are said to be isomorphic within vertices of degree 2 if they are isomorphic or if they can be transformed into isomorphic graphs by repeated insertions and/or removals of vertices of degree 2.

10.23 Kuratowski's Theorem

A graph is planar if and only if it does not contain any subgraph that is isomorphic to within vertices of degree 2 to either K_5 or $K_{3,3}$.

Remark : According to Kuratowski's theorem, there are numerous non planar graphs, they all must contain a subgraph that is isomorphic to within vertices of degree 2 to one of the kuratowski's graphs.

Example 10.37 *Draw a planar representation of each graphs for the following figures, if possible*

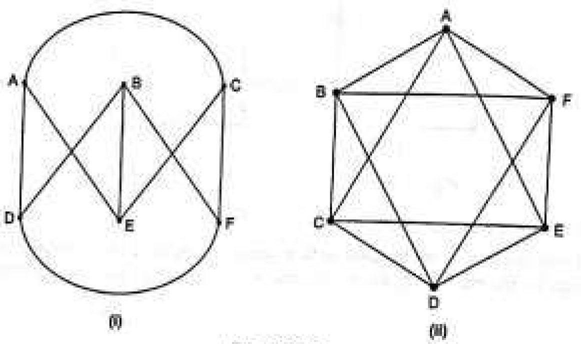


Fig. 10.125

Solution :

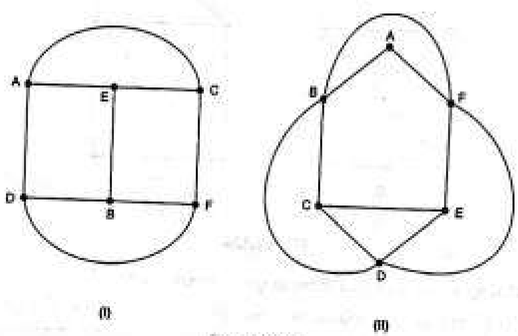


Fig. 10.126

(i) and (ii) are planar graphs.

Example 10.38 *Which of the following graphs represent bipartite graph and planar graph? If planar graph redraw the same.*

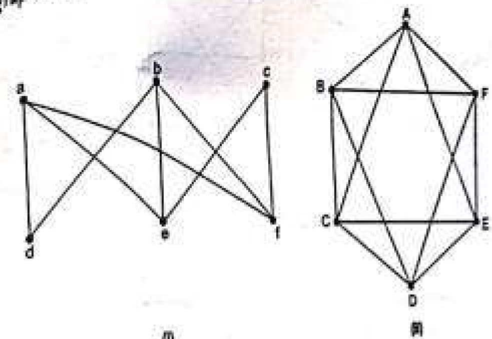


Fig. 10.127

Solution : (i) is a bipartite graph where

$V_1 = \{a, b, c\}$

$V_2 = \{d, e, f\}$

and $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ also vertices of V_1 joines with vertices of V_2. This is graph $K_{3,3}$ which is also complete and regular graph but this graph is not planar graph.

(ii) is not a bipartite graph as it can't be partitioned in to vertex sets V_1 and V_2. Such that vertices of V_1 joines with vertices of V_2.

This graph is a planar graph.

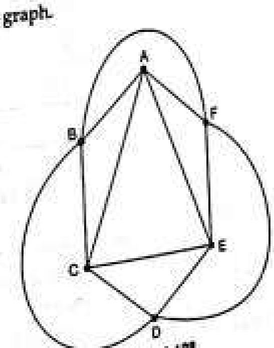


Fig. 10.128

Example 10.39 *Identify whether the graph given are planar or not.*

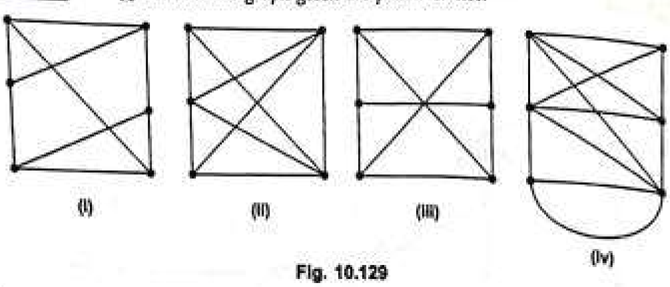


Fig. 10.129

Solution :

(i)

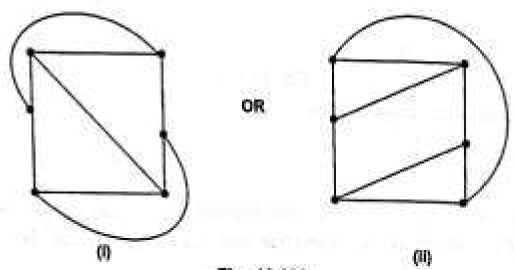
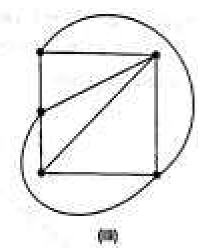


Fig. 10.130

(ii)



(iii) Not planar

(iv) Not planar.

Fig 10.131

Three Utility Problem

Example 10.40 *There are three houses H_1, H_2, H_3 each to be connected to each of the three utilities : Water (W), Gas (G) and Electricity (E) by means of conduits. Is it possible to make such connections without any crossovers of the conduits?*

Solution : The problem can be represented by a graph shown below in Fig. 10.132, the conduits are shown as edges while the houses and utility supply centres are vertices.

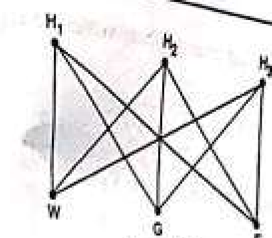


Fig. 10.132

The above graph is a complete bipartite graph $k_{3,3}$. Which is a non planar graph. Hence it is not possible to draw edges without crossing over.

Therefore it is not possible to make connection without any crossovers of the conduits.

Example 10.41 *Determine which of the graphs of the given figure represent Eulerian circuit, Hamiltonian circuit, Bipartite graphs and planar graph. Justify your answer.*

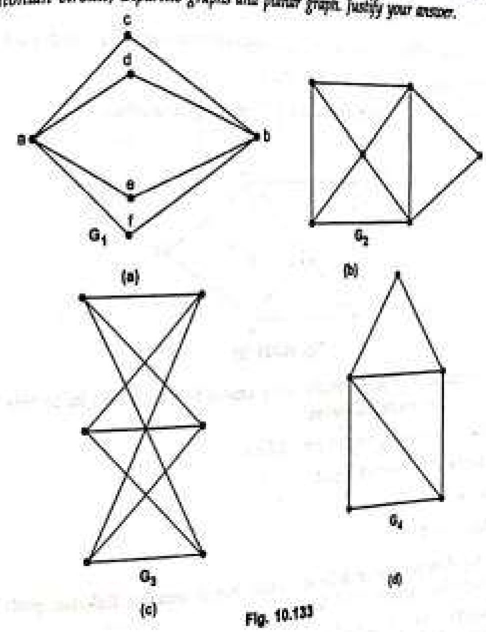


Fig. 10.133

Solution : (i) $G_1 = (V, E)$ is a bipartite graph as vertices of G_1 can be divided in V_1 and V_2

where $V_1 = \{a, b\}$, $V_2 = \{c, d, e, f\}$

such that $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \varnothing$.

Also edges of G_1 join the vertices from V_1 to V_2

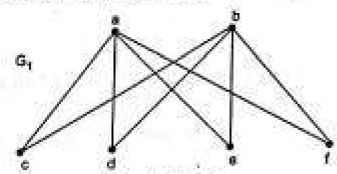


Fig. 10.134 (a)

Also it is a complete bipartite graph $K_{2,4}$.

Degree of each vertex is even hence G_1 contain Eulerian circuit a c b d a e b f a.

G_1 does not contain any Hamiltonian circuit.

G_1 is a planar graph as no two edges are intersecting each other.

(ii)

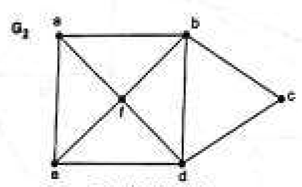


Fig. 10.134 (b)

It is not a bipartite graph as vertices of G_2 cannot be partitioned in to sets V_1 and V_2 not having edges among themselves.

G_2 contains a Hamiltonian circuit a b c d f e a.

In G_2 degree of two vertices is odd

deg (a) = 3

deg (e) = 3

Hence G_2 does not contain Eulerian circuit, but it contains Eulerian path as G_2 contains exactly two vertices of odd degree.

Eulerian path is a b c d f b d e f a e.

G_2 is a planar graph as no two edges are intersecting each other.

(iii)

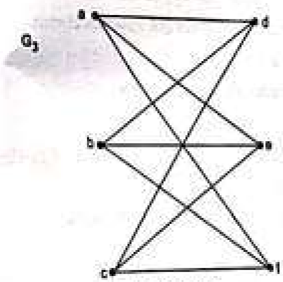


Fig. 10.134 (c)

G_3 is a bipartite graph. V = {a, b, c, d, e, f} can be partitioned in two parts V_1 and V_2

Such that $V_1 = \{a, b, c\}$, $V_2 = \{d, e, f\}$

and $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \varnothing$ and also edges of G_3 join the vertices from V_1 to V_2.

G_3 is a complete bipartite graph $K_{3,3}$. G_3 is also a regular graph as degree of each vertex is 3.

In G_3 degree of all vertices is odd hence G_3 does not contain Eulerian circuit as well as Eulerian path.

G_3 is a graph $K_{3,3}$, i.e. $K_{m,n}$ where m = n hence G_3 has a Hamiltonian circuit f c e a d b f.

$G_3 = K_{3,3}$ is known as Kuratowski's second graph which is a non planar graph.

(iv)

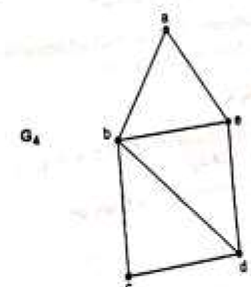


Fig. 10.134 (d)

G_4 is not a bipartite graph.

Degree of two vertices d and e are odd.

Hence G_4 contains Eulerian path but not Eulerian circuit.

Eulerian path is e a b e d b c d.

G_4 contains a Hamiltonian circuit a b c d e a.

G_4 is a planar graph.

Example 10.42 *Show that in a connected planar graph with 6 vertices and 12 edges, each of the regions is bounded by 3 edges.*

Solution : According to Euler's theorem for planar graph,

$$v - e + r = 2$$

where v is the number of vertices

e is the number of edges

and r is the number of regions

$$v = 6, e = 12$$

$$\Rightarrow \quad 6 - 12 + r = 2$$

$$\Rightarrow \quad r = 8$$

Hence, the given graph contains 8 regions. If each region is bounded by 3 edges then,

$$2e = 3r$$

i.e. $\quad 2 \times 12 = 3 \times 8$

$\Rightarrow$ 24 = 24, which is true

Hence each regions of the given graph is bounded by 3 edges.

10.24 Matrix Representation of Graph

A graph is completely determined by either its adjacencies or its incidences. This information can be stated in matrix form.

10.24.1 Adjacency Matrix

Let G = (V, E) be a simple digraph in which $V = \{v_1, v_2, \cdots v_n\}$ and the vertices are assumed to be ordered from v_1 to v_n.

An n × n matrix A whose elements a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if}(v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

is known as adjacency matrix of the graph G.

Example

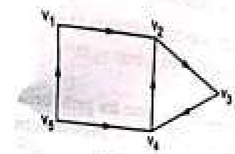


Fig. 10.135

Adjacency matrix A of graph G

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Remark : Number of 1's in A is equal to the number of edges in the graph.

10.24.2 Boolean Matrix

The binary digits or bits are the symbol 0 and 1. Consider the following operations on these digits.

+	0	1
0	0	1
1	1	1

×	0	1
0	0	0
1	0	1

If 0 represents false and 1 represents TRUE, the above operations correspond to the logical operations '∨' (OR) and '∧' (AND) respectively.

∨	F	T
F	F	T
T	T	T

∧	F	T
F	F	F
T	F	T

The above operations on 0 and 1 are called Boolean operations.

Let $A = [a_{ij}]$ be a matrix whose entries are the bits 0 and 1 subject to the above Boolean operations. Then A is known as Boolean matrix.

10.25 Warshall's Algorithm to Produce Shortest Path

Let G be a directed graph with m vertices $v_1, v_2, \ldots, v_m$. Suppose G is weighted, i.e. suppose each edge e of G is assigned a non-negative number w(e) called weight or length or e.

The weight matrix $W = (w_{ij})$ is defined by

$$w_{ij} = \begin{cases} w(e) & \text{if there is an edge e from } v_i \text{ to } v_j \\ 0 & \text{if there is no edge from } v_i \text{ to } v_j \end{cases}$$

The path matrix P tells, whether or not there are paths between the vertices. We want to find matrix D which tells us the lengths of the shortest paths between the vertices.

$\therefore$ $D = (a_{ij})$ where

a_{ij} = Length of the shortest path from v_i to v_j

Here we define a sequence of matrices $D_0, D_1, D_2, \ldots D_n$ (analogus to the above matrices $P_0, P_1, P_2, \ldots P_n$)

where ,

$D_k[i, j]$, The, ij entry of D_k is defined as

$D_k[i, j]$ = The smaller of the length of the preceding path from v_i to v_j or the sum of the lengths of the preceding paths from v_i to v_k and from v_k to v_j.

i.e.

$$D_k[i, j] = \text{MIN}\,(D_{k-1}(i, j), D_{k-1}(i, k) + D_{k-1}(k, j))$$

The initial matrix D_0 is the same as the weight matrix W except that 0 in W is replaced by ∞ (a very large number). The final matrix D_m will be the desired matrix D.

Example 10.43 *Find the shortest path between each pair of vertices for a simple digraph using Warshall's algorithm.*

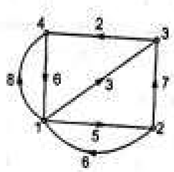


Fig. 10.136

Solution : Weight matrix,

$$W_{ij} = \begin{cases} W(e) & \text{if there in an edge e from } v_i \text{ to } v_j \\ 0, & \text{if there is not edge vrom } v_i \text{ to } v_j \end{cases}$$

1 2 3 4

$$W = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 0 & 5 & 3 & 8 \\ 6 & 0 & 7 & 0 \\ 0 & 0 & 0 & 2 \\ 6 & 0 & 0 & 0 \end{bmatrix}$$

Distance matrix D_0 is same as of weight matrix W except that 0 in W(other than diagonal) are replaced by ∞ (a very large number)

$$D_0 = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 5 & 3 & 8 \\ 2 & 6 & 0 & 7 & \infty \\ 3 & \infty & \infty & 0 & 2 \\ 4 & 6 & \infty & \infty & 0 \end{array}$$

In distance matrix D_1, keep first row and first column as D_0 and diagonal entries as zero. To fill other entries of D_1, apply.

$$\begin{aligned} D_1(2,3) &= \text{Min}\,[D_0(2,3), D_0(2,1)+D_0(1,3)] \\ &= \text{Min}\,[7, 6+3] \\ &= \text{Min}\,[7, 10] \\ &= 7 \\ D_1(2,4) &= \text{Min}\,[D_0(2,4), D_0(2,1)+D_0(1,4)] \\ &= \text{Min}\,[\infty, 6+8] \\ &= \text{Min}\,[\infty, 14] \\ &= 14 \\ D_1(3,2) &= \text{Min}\,[D_0(3,2), D_0(3,1)+D_0(1,2)] \\ &= \text{Min}\,[\infty, \infty+5] \\ &= \infty \\ D_1(3,4) &= \text{Min}\,[D_0(3,4), D_0(3,1)+D_0(1,4)] \\ &= \text{Min}\,[2, \infty+8] \\ &= 2 \\ D_1(4,2) &= \text{Min}\,[D_0(4,2), D_0(4,1)+D_0(1,2)] \\ &= \text{Min}\,[\infty, 6+5] \\ &= 11 \\ D_1(4,3) &= \text{Min}\,[D_0(4,3), D_0(4,1)+D_0(1,3)] \end{aligned}$$

$$= \text{Min}\,[\infty, 6 + 3]$$
$$= 9$$

$$D_1 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 5 & 3 & 8 \\ 6 & 0 & 7 & 14 \\ \infty & \infty & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

Similarly to find D_2 from D_1, keep second row and second column of D_1 as it is and diagonal entries as zero. To fill other entries apply,

$$D_2(1,3) = \text{Min}\,[D_1(1,3), D_1(1,2)+D_1(2,3)]$$
$$= \text{Min}\,[3, 5 + 7] = 3$$
$$D_2(1,4) = \text{Min}\,[D_1(1,4), D_1(1,2)+D_1(2,4)]$$
$$= \text{Min}\,[8, 5 + 14] = 8$$
$$D_2(3,1) = \text{Min}\,[D_1(3,1), D_1(3,2)+D_1(2,1)]$$
$$= \text{Min}\,[\infty, \infty + 6] = \infty$$
$$D_2(3,4) = \text{Min}\,[D_1(3,4), D_1(3,2)+D_1(2,4)]$$
$$= \text{Min}\,[2, \infty + 14] = 2$$
$$D_2(4,1) = \text{Min}\,[D_1(4,1), D_1(4,2)+D_1(2,1)]$$
$$= \text{Min}\,[6, 11 + 6] = 6$$
$$D_2(4,3) = \text{Min}\,[D_1(4,3), D_1(4,2)+D_1(2,3)]$$
$$= \text{Min}\,[9, 11 + 7] = 9$$

$$D_2 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 5 & 3 & 8 \\ 6 & 0 & 7 & 14 \\ \infty & \infty & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

Similarly to find D_3 from D_2, keep third row and third column of D_2 unchanged and diagonal entries as zero. To find other entries apply

$$D_3(1,2) = \text{Min}\,[D_2(1,2), D_2(1,3)+D_2(3,2)]$$

$$= \text{Min}\,[5, 3 + \infty] = 5$$
$$D_3(1,4) = \text{Min}\,[D_2(1,4), D_2(1,3)+D_2(3,4)]$$
$$= \text{Min}\,[8, 3 + 2] = 5$$
$$D_3(2,1) = \text{Min}\,[D_2(2,1), D_2(2,3)+D_2(3,1)]$$
$$= \text{Min}\,[6, 7 + \infty] = 6$$
$$D_3(2,4) = \text{Min}\,[D_2(2,4), D_2(2,3)+D_2(3,4)]$$
$$= \text{Min}\,[14, 7 + 2] = 9$$
$$D_3(4,1) = \text{Min}\,[D_2(4,1), D_2(4,3)+D_2(3,1)]$$
$$= \text{Min}\,[6, 9 + \infty] = 6$$
$$D_3(4,2) = \text{Min}\,[D_2(4,2), D_2(4,3)+D_2(3,2)]$$
$$= \text{Min}\,[11, 9 + \infty] = 11$$

$$D_3 = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 5 & 3 & 5 \\ 6 & 0 & 7 & 9 \\ \infty & \infty & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

To find D_4 from D_3, keep fourth row and fourth column of D_3 unchanged and diagonal entries as zero. To find other entries apply

$$D_4(1,2) = \text{Min}\,[D_3(1,2), D_3(1,4)+D_3(4,2)]$$
$$= \text{Min}\,[5, 5 + 11] = 5$$
$$D_4(1,3) = \text{Min}\,[D_3(1,3), D_3(1,4)+D_3(4,3)]$$
$$= \text{Min}\,[3, 5 + 9] = 3$$
$$D_4(2,1) = \text{Min}\,[D_3(2,1), D_3(2,4)+D_3(4,1)]$$
$$= \text{Min}\,[6, 9 + 6] = 6$$
$$D_4(2,3) = \text{Min}\,[D_3(2,3), D_3(2,4)+D_3(4,3)]$$
$$= \text{Min}\,[7, 9 + 9] = 7$$
$$D_4(3,1) = \text{Min}\,[D_3(3,1), D_3(3,4)+D_3(4,1)]$$
$$= \text{Min}\,[\infty, 2 + 6] = 8$$
$$D_4(3,2) = \text{Min}\,[D_3(3,2), D_3(3,4)+D_3(4,2)]$$
$$= \text{Min}\,[\infty, 2 + 11] = 13$$

$$D_4 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 5 & 3 & 5 \\ 6 & 0 & 7 & 9 \\ 8 & 13 & 0 & 2 \\ 6 & 11 & 9 & 0 \end{bmatrix} \end{array}$$

Now D_4 is the matrix of shortest path between the vertices.

11 Trees

Syllabus

Definition, branch nodes, leaf (terminal) nodes, root, examples; Different representations of a tree, examples; Binary tree, m-ary tree, Full (or complete) binary tree, examples; Converting any m-ary tree to a binary tree, examples; Representation of a binary tree : Linked-list; Tree traversal : Pre-order, in-order, post-order traversal, examples, algorithms; Applications of List structures and graphs.

Contents

11.1 Definition of Tree

A graph is acyclic if it has no cycles (or circuits).

A **tree** is a **connected acyclic graph**. In other words a simple connected graph without circuits is known as tree.

Examples

1.

Fig. 11.1

2.

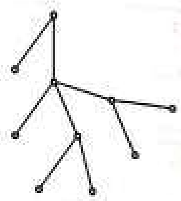


Fig. 11.2

3.

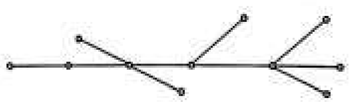


Fig. 11.3

4.

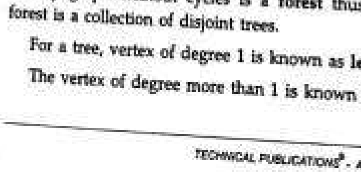
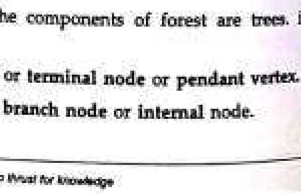


Fig. 11.4

5.



Is not tree as it contains a cycle 678.

Fig. 11.5

Any graph without cycles is a **forest** thus the components of forest are trees. i.e. forest is a collection of disjoint trees.

For a tree, vertex of degree 1 is known as **leaf** or **terminal node** or **pendant vertex**. The vertex of degree more than 1 is known as **branch node** or **internal** node.

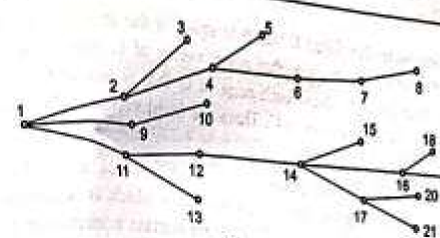


Fig. 11.6

Vertices 3, 5, 8, 10, 13, 15, 18, 19, 20, 21 are leaves and 1, 2, 4, 6, 7, 9, 11, 12, 14, 16, 17 are branch nodes.

Remark

1. Every tree is a bipartite graph.
2. Every tree with only countably many vertices is a planar graph.

11.1.1 Properties of Trees

Property 1

G is a tree if and only if there exists a **unique** path between every two vertices of G.

Proof

Since there exist a path between every two vertices of G this shows that G is a connected graph. Also G cannot contain a circuit as these paths are unique hence G is a connected graph without circuit. Therefore G is a tree.

Conversely, if the graph G is a tree, it is connected and hence, there must be at least one path between every pair of vertices in G.

Suppose there are two distinct path P_1 and P_2 between vertices a and b of G. The union of these paths will contain a circuit and then G will not be a tree, which is a contradiction.

Therefore, if G is a tree then it has a unique path between every pair of vertices of G.

Property II

G is a tree if and only if G is connected and has exactly (n-1) edges where n is number of vertices of G.

Proof

Given that G is a connected graph with total number of edges e = (n-1). For tree, it is sufficient to prove that it is noncyclic. Suppose G contains a circuit C. Let P denote the number of vertices in the circuit C which is equal to the number of edges in C also. By the connectivity of G, we can say that every vertex of G that is not in C must be connected to the vertices in C. Now each edge of G that is not in C can connect only one additional vertex to the vertices in C. There are (n-p) vertices of G which are not in the circuit C.

Thus G must contain at least (n-p) edges which are not in C. Hence, the total number of edges e in G is given by $e \geq P+(n-p)=n$ which is a contradiction to the hypothesis that e = n-1. It follows that G does not contain a circuit and is therefore a tree.

Conversely, if G is a tree then by definition it is connected and noncyclic. We have to prove that the number of edges in G are e = n-1. We will prove this result by induction on n. For n = 1 the result is obvious. Let n > 1, consider G - e', for any edge e' of G. Since a tree is a graph without circuits hence G - e' is a disconnected graph with two components G_1 and G_2.

Now G_1 and G_2 are connected and contain no cycles since they are both subgraphs of G which is noncyclic. Let G_i contains v_i vertices and e_i edges for $i = 1, 2$. By induction

$$e_i = n_i - 1, \quad i = 1, 2$$

Therefore, the number of edges in G is given by,

$$e = e_1 + e_2 + 1$$

$$\Rightarrow \quad e = (n_1 - 1) + (n_2 - 1) + 1$$

$$\Rightarrow \quad e = n_1 + n_2 - 1$$

$$\Rightarrow \quad e = n - 1$$

Hence a tree contains (n-1) number of edges. The result of the preceding theorems can be summarized by saying that the following are equivalent definitions of a tree. That is, a graph G with n vertices is called a tree, if

i) Every two vertices of G are joined by a unique path.

ii) G is a connected graph and e = n-1 where n is number of vertices and e is number of edges.

(i.e. the number of vertices is one more than number of edges in a tree).

iii) G is acyclic (without circuit) and n = e + 1.

iv) G is acyclic and if any two non adjacent vertices of G are joined by an edge x then G + x has exactly one cyclic (circuit).

v) G is connected and, if any edge is removed from G, then it is not connected anymore.

11.2 Eccentricity of a Vertex

Distance between two vertices in a connected graph is a length of the shortest path between them.

The eccentricity e(v) of a vertex v in a connected graph G is max. d (u, v) of all u in G. i.e. e(v) of a vertex v in a connected graph G is the distance from v to the vertex farthest from v in G.

$\boxed{e(v) = \max.\ d(u, v)}$ of all u in G

Example

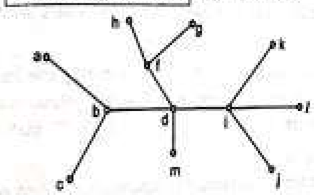


Fig. 11.7

e(a) = 4 (as h or g or k or l or j are farthest from a and distance between a and any one of them is 4 (i.e. there are 3 more vertices in between))

e(b) = 3

e(c) = 4

e(d) = 2

e(m) = 3

e(i) = 3

e(l) = 4

e(j) = 4

e(k) = 4

e(g) = 4

e(h) = 4

e(f) = 3

11.2.1 Center of a Tree

A vertex with minimum eccentricity is called a centre of G.

For the above Fig. 11.7, d is the centre of the tree as d has minimum eccentricity 2.

11.2.1.1 Theorem

Every tree has a centre consisting of either one vertex or two adjacent vertices.

Proof

The result is obvious for the trees with one vertex (i.e. K_1) and with two vertices and one edge (i.e. K_2).

We show that any other tree T has the same central vertices as the tree T', where T' is a tree obtained by removing all the end vertices of T. Clearly, the maximum of the distance from a given vertex u of T to any other vertex v of T will occur only when v is an end vertex.

Thus, the eccentricity of each vertex in T' will be exactly one less than the eccentricity of the same vertex in T. Hence the vertices of T which possess minimum eccentricity in T are the same vertices having minimum eccentricity in T', that is, T and T' have the same centre.

If the process of removing end vertices is repeated, we obtain successive trees having the same centre as T. Since T is finite, we eventually obtain a tree which is either K_1 or K_2.

In either case all vertices of this ultimate tree constitute the centre of T which thus consists of just a single vertex or of two adjacent vertices.

Example 1 :

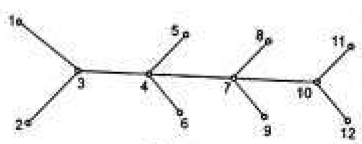


Fig. 11.8

e(4) = 3, e(7) = 3. Eccentricity of 4 and 7 are minimum.

Hence 4 and 7 are the two centres.

Example 2 :

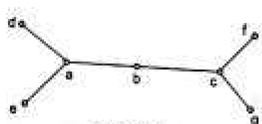


Fig. 11.9

e(b) = 2. Eccentricity of b is minimum. Hence b is the centre of the tree.

11.3 Cut Points (or Cut Vertices)

In the topic graph theory, we have already defined the cut points as those vertices whose deletion disconnects the graph.

In any tree, all the vertices except pendant vertices (or leaves) are cut vertices.

Example :

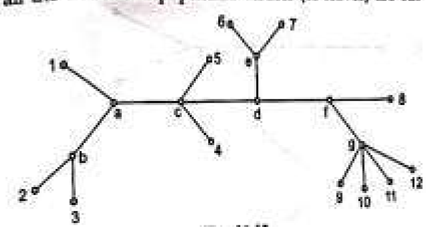


Fig. 11.10

For the above tree in Fig. 11.10 a, b, c, d, e, f, g are cut vertices and 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 are pendant vertices.

For a non trivial tree

$$\sum_{i=1}^{n} \deg(v_i) = 2e = 2(n-1)$$

has to divide among n vertices.

Since no vertex can be of zero degree (as tree is connected graph) there are at least two vertices with degree less than 2.

i.e. degree of at least 2 vertices is one. Hence in any tree, at least two pendant vertices are present.

Hence result : Every nontrivial tree has at least two pendant vertices (or end vertices).

Example 11.1 *Which trees are complete bipartite graphs ?*

Solution : Let T be a tree which is a complete bipartite graph. Let $T = K_{m,n}$, then the total number of vertices in T is (m + n). Hence the tree contains (m + n − 1) number of edges. But the graph $K_{m,n}$ has (mn) number of edges.

Therefore,

$$m + n - 1 = mn$$

$$\Rightarrow \quad (m-1)(n-1) = 0$$

$$\Rightarrow \quad m = 1 \text{ or } n = 1$$

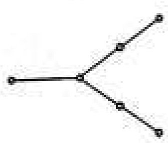
This means T is either $K_{1,n}$ or $K_{m,1}$ i.e. T is a star.

Example 11.2 *Draw all nonisomorphic trees with six points.*

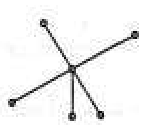
Solution :

1) 

2) 

3) 

4) 

5) 

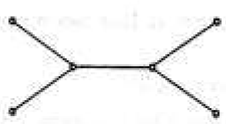
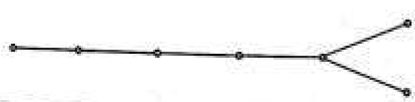
6) 

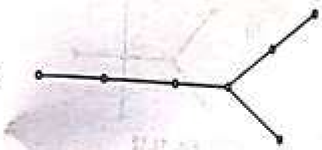
Fig. 11.11

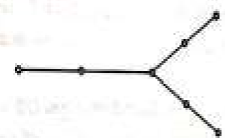
Example 11.3 *Draw all nonisomorphic trees with seven points.*

Solution :

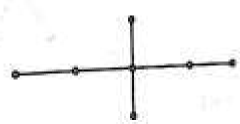
1) 

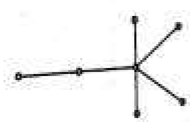
2) 

3) 

4) 

5) 

6) 

7) 

8) 

9) 

10) 

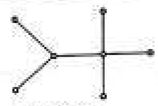
11) 

Fig. 11.12

Example 11.4 *Show that it is possible to draw a tree with 10 vertices which has vertices either of degree 1 or of degree 3. Draw the tree. Is it possible to draw the same type of tree with 11 vertices ?*

Solution : Given the tree has 10 vertices. Therefore it has (10 − 1), i.e. 9 edges. Suppose there are x number of vertices of degree 1 and y number of vertices of degree 3 in the tree.

Hence $x + y = 10$

also $2e = \sum_{i=1}^{n} d(v_i)$

$\Rightarrow \quad 2 \times 9 = x + 3y$

$\Rightarrow \quad 18 = x + 3y$

$\Rightarrow \quad x = 6$ and $y = 4$

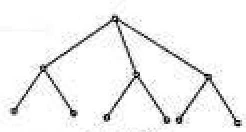


Fig. 11.13

Now if number of vertices are 11

then $e = 10$

hence $2e = \sum_{i=1}^{n} d(v_i)$

$\Rightarrow \quad 20 = x + 3y$

hence $x = \frac{13}{2}, \; y = \frac{9}{2}$ which is not possible.

Therefore, it is not possible to draw a tree with 11 vertices which has vertices of degree 1 or 3.

11.4 Rooted Tree and Binary Tree

A directed graph is said to be a directed tree if it becomes a tree when the directions of the edges are ignored.

Example

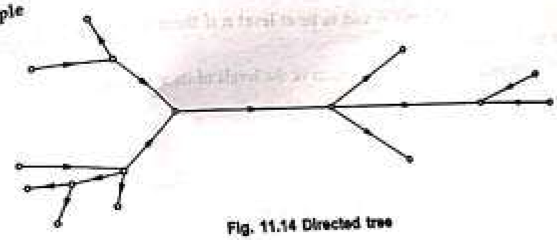


Fig. 11.14 Directed tree

Tree in Fig. 11.14 is a directed tree.

A directed tree is called a rooted tree if there is exactly one vertex whose incoming degree is zero and the incoming degrees of all other vertices are one.

The vertex with incoming degree 0 is called the root of the rooted tree.

Example of rooted tree.

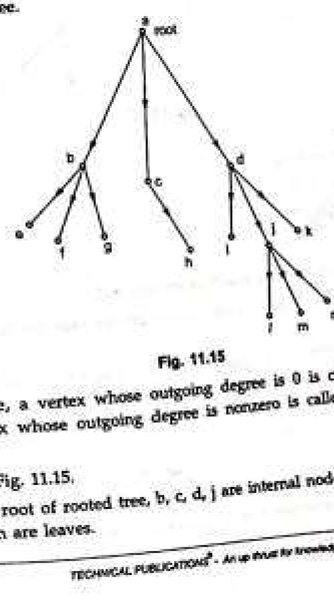


Fig. 11.15

In a rooted tree, a vertex whose outgoing degree is 0 is called leaf or a terminal node, and a vertex whose outgoing degree is nonzero is called a branch node or an internal node.

For the above Fig. 11.15.

a is known as root of rooted tree, b, c, d, j are internal nodes or branch nodes and e, f, g, h, i, k, *l*, m, n are leaves.

11.4.1 Level and Height of a Tree

A vertex v in a rooted tree is said to be at **level n** if there is a path of length n from the root to the vertex v.

The **height** of the tree is the maximum of the levels of its vertices.

Example

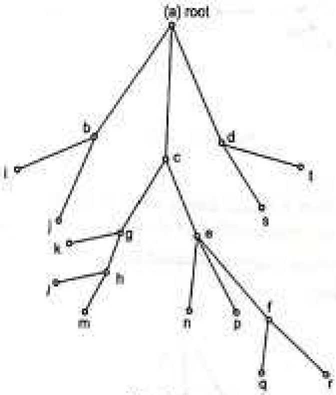


Fig. 11.16

For the Fig. 11.16

Vertices b, c, d are at level 1

i, j, g, e, s and t are at level 2

k, h, n, p, f are at level 3

l, m, q, r at level 4.

The maximum level is 4, hence height of the tree is 4.

In a rooted tree, if the level of a vertex u is greater than the level of the vertex v, then u is below v.

If u is below v and there is an edge from v to u, then u is said to be son of v (or child of v) and v is said to be the father of u (or parent of u). Two vertices are said to be brothers, if they are sons of the same vertex.

A leaf is a vertex without children.

If $V = \{a, v_1, v_2, v_3, \ldots, v_{n-1}, b\}$ is a path from a to b, then b is known as descendant of a and a is called as ancestor of b.

These terms remind us that family trees are rooted trees.

Example

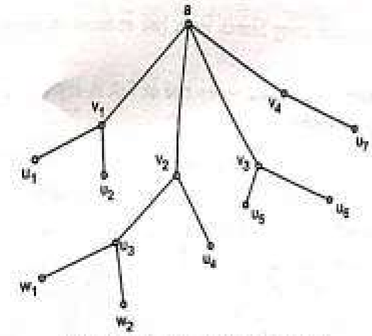


Fig. 11.17 'a' is root of rooted tree

For the Fig. 11.17 a is root of rooted tree. 'a' is father of v_1, v_2, v_3 and v_4 or in other words v_1, v_2, v_3, v_4 are sons of a. Also v_1, v_2, v_3, v_4 are brothers.

u_1, u_2 are sons of v_1 or v_1 is father of u_1 and u_2 also u_1, u_2 brothers.

Similarly u_3, u_4 are brothers and u_3, u_4 are sons of v_2 also u_5, u_6 are brothers and son of v_3. v_4 is father of u_7.

w_1, w_2 are sons of u_3.

w_1, w_2 are known as descendant of v_2 as well as descendant of a. Similarly $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ are descendant of a.

'a' is ancestor of $u_1, u_2, u_3, u_4, u_5, u_6, u_7$ and also ancester of w_1, w_2.

v_2 is ancestor of w_1 and w_2.

Remark

A tree without any designated root is called a **free tree** or simple tree.

11.4.2 Ordered Tree

An ordered tree is a rooted tree for which an ordering is specified for the children (or sons) of each vertex.

11.4.3 Subtrees

A subtree of a tree T is a tree comprised of a node in T and all of its descendants in T.

The subtree corresponding to the root node is the entire tree, the subtree corresponding to any other node is called a proper subtree.

11.4.4 M-ary Trees

An ordered tree in which every branch node has at most m sons is known as m-ary tree.

An m-ary tree is known as **regular m-ary tree** or **full m-ary tree** if every branch node has exactly m sons or zero sons.

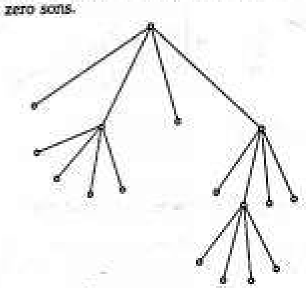


Fig. 11.18 4-ary tree

Fig. 11.18 represent a regular 4-ary tree.

While Fig. 11.19 is 3-ary tree which is not regular.

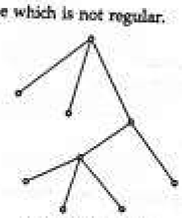


Fig. 11.19 3-ary tree

11.4.4.1 Theorem

A regular m-ary tree with k interior nodes (branch nodes) has (mk+1) nodes at all.

Proof

Suppose the given regular m-ary tree has n number of vertices, out of which there are k branch nodes. Clearly, the number of leaves or sons in the tree is

$$t = (n - k)$$

Since the given graph is a regular m-ary tree.

Hence each of the k branch nodes has m sons and thus the regular m-ary tree will have total (mk) sons. But root is not a son. Therefore the given tree has a total of (mk+1) nodes.

Hence $\boxed{n = mk + 1}$

11.5 Binary Trees

An important class of m-ary tree is binary trees.

An m-ary tree is known as binary tree if every branch node has at most 2 sons.

A binary tree is known as regular binary tree or full binary tree if every branch node has exactly 2 sons or zero sons.

Example 1 : The tree in the following Fig. 11.20 is a binary tree.

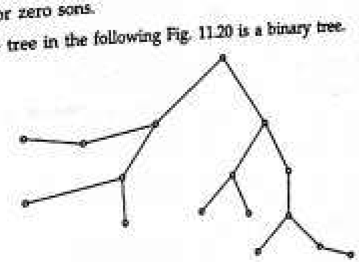


Fig. 11.20 Binary tree

Example 2 : The tree in the following Fig. 11.21 is a regular binary tree (or full binary tree).

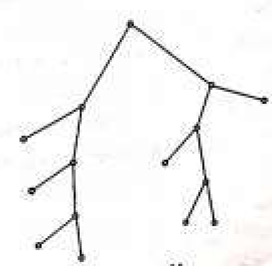


Fig. 11.21

For binary trees, instead of referring to the first subtree or the second subtree of a branch node, we often refer to the left subtree or the right subtree of the node.

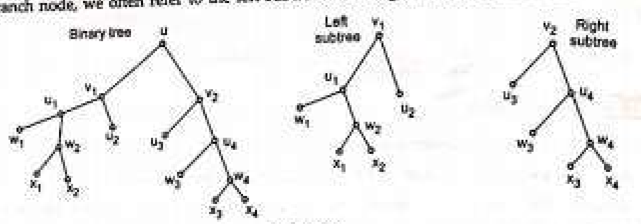


Fig. 11.22

Remark

The number of vertices in a full binary tree with h levels is

$1+2+2^2+2^3+.......+2^h$ and this is a geometric progression hence the sum of (h + 1) terms

$$S_{h+1} = 1+2+2^2+2^3+........+2^h \quad ... (1)$$

$$2(S_{h+1}) = 2+2^2+2^3+.....2^h+2^{h+1} \quad ... (2)$$

Subtracting equation (1) from equation (2)

$$S_{h+1}(2-1) = 2^{h+1}-1$$

or $$S_{h+1} = 2^{h+1}-1$$

Example 11.5 *How many leaves does a full binary tree with h levels have ?*

Solution : A full binary tree with h levels has 2^h leaves.

Example 11.6 *How many leaves does a full binary tree with 25 levels have ?*

Solution : A full binary tree with 25 levels has 2^{25} leaves.

Example 11.7 *How many internal vertices does a full binary tree with h levels have?*

Solution : The total number of vertices in full binary tree with h levels

$$= 2^{h+1}-1$$

Out of which 2^h are leaves.

Hence internal vertices $= (2^{h+1}-1)-2^h$

$$= 2^h(2-1)-1 = 2^h-1$$

Example 11.8 *How many vertices will a full ternary tree with h levels have ?*

Solution : The number of vertices in a full ternary tree with h levels is

$$S = 1+3+3^2+3^3+........+3^h$$

$$\Rightarrow \quad S = \frac{3^{h+1}-1}{3-1}$$

i.e. $$S = \frac{3^{h+1}-1}{2}$$

Hence total number of vertices $= \frac{3^{h+1}-1}{2}$

Example 11.9 *Suppose you have a full ternary tree with 25 levels*

i) What level is vertex 50 in ?

ii) What are the labels of the three children of vertex 50 ?

iii) What is the label of the parent of vertex 50 ?

Solution : i) The last vertex on level 1 is vertex 4 $(1+3^1)$ (if root is considered as vertex 1), the last vertex of the level 2 is vertex $(1+3^1+3^2)$ 13, and in general the last vertex on level h is vertex

$$(1+3+3^2+....+3^h) = \frac{1}{2}(3^{h+1}-1)$$

Substituting h = 3, gives that the vertex 40 is the last vertex on level 3 and similarly the last vertex on level 4 is vertex 121. Therefore vertex 50 is on level 4.

ii) The children of vertex 41 are vertices 122, 123, 124 those of vertex 42 are vertices 125, 126, 127. We can see that if vertex 41 + x is on level 4, its children are labelled 122 + 3x, 123 + 3x and 124 + 3x.

In particular the children of vertex 50, (which is 41 + 9), are labelled 149, 150, 151.

iii) If we do the same on level 3 (vertices 14 to 40) we have that the children of vertex 14 are 41, 42, 43 and for vertex 15 (i.e. 14 + 1) are 41 + 3.1, 42 + 3.1, 43 + 3.1 i.e. children of 15 are 44, 45, 46 in similar way the children of 14 + x are 41 + 3.x, 42+3.x, 43 + 3.x and 50 can be written as 41 + 3.3.

Hence 14 + 3, i.e. vertex 17 is the parent vertex of 50.

Example 11.10 *A telephone network is established among 100 people. Information received by the first person is passed along to the 99 others as follows ; the first person calls exactly 3 people, and each of these people call 3 others and so on till there are no others to call. If each call taken 5 minutes, then how long does it take for message to be relayed from first person to receive the message to every one else ? How many people make no calls ?*

Solution :

The given situation can be represented using a rooted tree with 100 nodes where each node represents a person and root node is the first person.

Now first person calls exactly 3 persons hence root node has 3 sons, also each these sons call another 3 persons. In short this is a full ternary tree.

Hence from the result

$$n = mi + 1$$

$m = 3,\quad n = 100$

Hence $\quad 100 = 3i + 1$

$\Rightarrow \quad i = 33.$

Hence there are 33 interior nodes, hence 100 − 33 = 67 are the leaf nodes, and therefore 67 people do not make any call.

Now each call takes 5 minutes.

Hence, after 5 minutes $1+3^1 = 4$ people have received the message (1 is for the root node).

Each of these 3 people will call 3 other people.

Hence after 5 + 5, i.e. 10 minutes.

$1+3^1+3^2$ i.e. 13 people have received the message.

Similarly, after 15 minutes $1+3^1+3^2+3^3 = 40$ people have received the message.

After 20 minutes (i.e. 4.5 minutes)

$1+3^1+3^2+3^3+3^4 = 121$ people

But we have to pass message to 100 people.

Hence within 20 minutes 100 people will receive the message.

Example 11.11 *What is the total number of nodes in a full binary tree with 20 leaves ?*

Solution : Let n represent the total number of nodes in a full binary tree.

Then by the result

$$n = mi + 1$$

$$m = 2$$

Hence $\quad n = 2i + 1$

where, i is the number of branch nodes.

Total number of nodes in a tree are the sum of branch nodes and leaves.

Hence $\quad n = i + 20$

or $\quad i = n - 20$

Hence $\quad n = 2(n - 20) + 1$

or $\quad n = 39$

Hence total number of nodes in a full binary tree with 20 leaves are 39.

Example 11.12 *If there are 60 contestants in a single elimination tournament, how many matches are played ?*

Solution : A single elimination tournament can be represented by a full binary tree, in which contestant represent terminal vertices and winners represent the internal vertices. According to the problem, there are 60 contestants (leaves). Let i be the number of internal vertices.

Hence the total number of vertices

$$n = \text{Internal vertices} + \text{Leaves}$$

Therefore $\quad n = i + 60$

also from the result

$$n = mi + 1$$

$$m = 2$$

Hence $\quad n = 2i + 1$

Therefore $\quad 2i + 1 = i + 60$

Hence $\quad i = 59$

Hence 59 matches are played in the tournament.

Example 11.13 *Can we have a ternary tree with exactly 21 nodes ?*

Solution : From the result

$$n = mi + 1$$

$n = 21,\ m = 3$

$$21 = 3i + 1$$

or $\quad 3i = 20$

Hence i is not a whole number. Therefore such tree is not possible.

Example 11.14 *19 lamps are to be connected to a single electrical outlet, using extension chords, each of which has 4 outlets, find the number of extension chords needed and draw the corresponding tree.*

Solution : Represent lamps by the leaves and the extension chords by the branch nodes of the tree. Since there are 19 lamps to be connected and each extension chord has 4 outlets, this means the tree has 19 leaves and each branch node has 4 sons. Hence the tree is a full 4-ary (quaternary) tree with 19 leaves.

Now, the total number of vertices in a full m-ary tree is

$n = mi + 1$, where i is the number of branch nodes.

Hence $n = 4i + 1$ as the tree is full 4-ary tree.

Here $\quad n = 19 + i$

Hence $\quad 4i + 1 = 19 + i$

$\Rightarrow \quad i = 6$

Hence, 6 extension chords are required to connect 19 lamps with a single outlet. The required tree is shown below in Fig. 11.23.

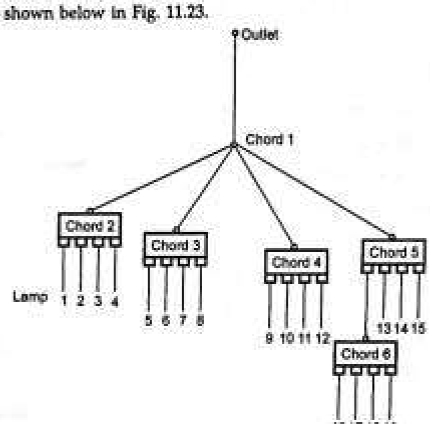


Fig. 11.23

11.6 Prefix Code

A code is simply an organized way of representing information using a fixed, possibly small set of symbols.

For example, modern computers rely on a binary (or two element) code consisting only of the symbols 0 and 1. All the information on computer (text, images, computer programs etc.) is stored in coded form as a sequence of 0s and 1s.

For example

If we use ASCII code then the letters H, E, L, O are represented as

Letter	Code
H	1001000
E	1000101
L	1001100
O	1001111

Based on the above information **HELLO** would be stored on a computer in an ASCII code string as

10010001000101100110010011001001111

which is longer than they need to be.

All the ASCII code words are of the same length (7 binary digits for the standard ASCII and 8 binary digits for extended ASCII).

Now the word in a code don't have to be all of the same length. However, by making each code word contain a fixed number of binary digits we ensure that an important property called the prefix property holds true for the ASCII code.

11.6.1 Codes and the Prefix Property

Any code used for transmitting information must possess prefix property. If a code has prefix property, it means that **no word in the code is a prefix of any other word in the code.**

In other words, that no longer code words begin with the identical digits making up any of the shorter code words.

As an example, let's consider a code made of code words of varying length. If 0 is a valid code word, then no other code word can start with 0 if we wish to preserve the prefix property.

If 10 is a valid code word, then no other code word can start with 10.

Therefore, a valid five word 'prefix code' containing these two code words might look like one of these codes.

Letter	Code 1	Code 2
A	0	1
B	10	00

C	110	011
D	1110	0100
E	1111	0101

The following codes, on the other hand, would be invalid because the prefix property does not hold.

Letter	Code 3	Code 4
A	1	0
B	00	10
C	01	01
D	001	011
E	011	100

In code 3, the code for B is a prefix of the code for D, the code for C is a prefix of the code for E.

In code 4, the code for A is a prefix of codes for C and D, the code for C is a prefix of the code for D.

11.6.2 The Prefix Property and Binary Trees

An intuitive way of discovering whether or not the prefix property holds is by drawing a code tree.

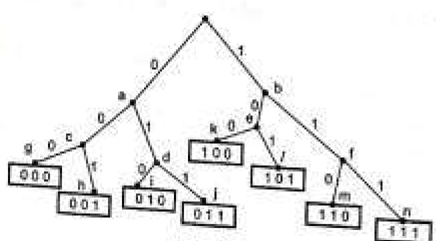


Fig. 11.24

The above Fig. 11.24 shows a binary code tree with 3 levels. We can build code words by starting at the top of the (upside-down) tree with an empty code word string " ", and descending through the tree. At each branch point, we choose to go either left (digit 0) or right (digit 1) and we add this new digit to the end of our code word string.

a, b, c, d, e, f are known as interior nodes or branch nodes.

and g, h, i, j, k, l, m, n are leaves.

In a prefix code, no code word can be a prefix of any other code word.

Hence, **if we draw a code tree for prefix code, all code words will be located at leaf nodes in that tree.**

11.6.3 For Every Prefix Code there is a Binary Tree

We want to show that for each prefix code there is a binary tree with the two edges incident from each of the branch nodes labeled with 0 and 1, such that the sequences of 0s and 1s assigned to the leaves are the sequences in the code. Let h denote the length of the longest sequences in the prefix code. We construct a full regular binary tree of height h and label the two edges incident from each of the branch nodes with 0 and 1. Let us assign to each vertex a sequence of 0s and 1s i.e. the sequence of labels of the edges in the path from the root to the vertex. Clearly, each binary sequence of length less than or equal to h is assigned to exactly one vertex. Let us mark all the vertices that are assigned with sequences in the prefix code, and then prove the tree by removing every vertex, together with the edge incident to it, that does not have a marked descendant. Since no marked vertex is an ancestor of another marked vertex, the set of marked vertices will be exactly the set of leaves of the resultant tree. This is, indeed, the tree we are looking for.

11.6.4 Optimal Trees

A binary tree with t leaves of weights $W_1, W_2, \ldots\ldots, W_t$.

s.t. $W_1 \leq W_2 \leq W_3 \leq \ldots \leq W_t$

then the weight of a binary tree W(T) is defined as

$$W(T) = \sum_{i=1}^{t} W_i \, l(W_i)$$

where $l(W_i)$ is the path length of the leaf to which the weight W_i is assigned.

A binary tree T for the weights $W_1, W_2, \ldots W_t$ is said to be an optimal tree if W(T) is minimum.

11.6.5 Huffman Coding

In Computer Science and Information theory, Huffman coding is an entropy encoding algorithm used for lossless data compression.

The term refers to the use of a variable length code table for encoding a source symbol (such as a character in a file) where the variable length code table has been derived in a particular way based on estimated probability of occurrence for each possible value of the source symbol.

It was developed by David A. Huffman while he was a Ph. D. student at MIT.

Huffman coding uses a specific method for choosing the representation for each symbol, resulting in a prefix code that expresses the most common characters using shorter string of bits than are used for less common source symbols.

11.6.6 Huffman Algorithm for Finding an Optimal Tree

Let W_1, W_2, W_3, ... be weights of the leaves and it is required to construct an optimal binary tree.

The following algorithm gives the required optimal binary tree.

Step 1 Arrange the weights in increasing order.

Step 2 Consider two leaves with minimum weights W_1 and W_2. Replace these two leaves and their father by a leaf and assign to this new leaf the weight $W_1 + W_2$.

Step 3 Repeat step 2, for weights $(W_1 + W_2)$, W_3, W_4, ... until no weight remain.

The tree obtained in this way is an optimal tree for given weights.

11.6.7 Optimal Prefix Code

A binary prefix code obtained from an optimal tree is called an **optimal prefix code**.

11.7 Binary Search Trees

In Computer Science, a Binary Search Tree (BST) is a binary tree which has the following properties.

1. Each node has a value.
2. A total order is defined on these values.
3. The left subtree of a node contains only values less than the node's value.
4. The right subtree of a node contains only values greater than or equal to the node's value.

Binary search trees are a fundamental data structure used to construct more abstract data structures such as sets, multisets and associative arrays (closely related to the mathematical concept of a function with a finite domain. As a consequence a common and important use of associative arrays is in memorization).

If a BST allows duplicate values, then it represents a multiset. This kind of tree uses non-strict inequalities. Everything in the left subtree of a node is strictly less than the

value of the node, but everything in the right subtree is either greater than or equal to the value of the node.

If a BST doesn't allow duplicate values, then the tree represents a set with unique values, like the mathematical set. Trees without duplicate values use strict inequalities, meaning that the left subtree of a node only contains nodes with values that are less than the value of the node, and the right subtree only contains values that are greater.

Let us now formulate precisely the problem of searching for an item in an ordered list. We assume that we are dealing with items over which there is a linear ordering <.

Let $k_1, k_2, \ldots, k_n$ be the n items in an ordered list which are known as the keys.

Assume that $k_1 < k_2 < \ldots < k_n$.

Given an item x, our problem is to search the keys and determine whether x is equal to one of the keys or whether x falls between keys k_i and k_{i+1} for some i.

A search procedure consists of a sequence of comparisons between x and the keys where each comparison tells us if

i) $x = k_i$

ii) $x < k_i$

iii) $x > k_i$

We show now how we can describe a search procedure using a binary tree representation.

We define a search tree for the keys $k_1, k_2, \ldots, k_n$ to be a binary tree with n branch nodes and n+1 leaves.

The branch nodes are labelled $k_1, k_2, \ldots, k_n$ and the leaves are labelled $k_0, k_1, k_2, \ldots, k_n$ such that, for the branch node with the label k_i, its left subtree contains only vertices with labels k_j, $j < i$ and its right subtree contains only vertices with labels k_j, $j \geq i$.

Starting with root of the search tree, we compare a given item x with label of root k_i.

If x is equal to k_i, search is completed. If x is less than k_i, we compare x with left son of the root.

If x is greater than k_i, we compare x with right son of the root.

Such comparison continues till we find a match for x or till a leaf is reached. If a leaf labelled k_j is reached, it means that x is larger than the key k_j but less than the key k_{j+1}. If leaf labelled k_0 is reached, it means that $x < k_1$.

If leaf labelled k_n is reached, it means $x > k_n$.

Huffman coding

Example 11.15 *Build the Huffman tree for the following frequencies of six letters.*

E	29
I	5
O	7
P	12
S	4
T	8

Solution : To build the Huffman tree, we sort the frequencies into increasing order (4, 5, 7, 8, 12, 29).

S	4
I	5
O	7
T	8
P	12
E	29

Then we choose the two smallest values S and I (4 and 5) and construct a binary tree with labelled edges.

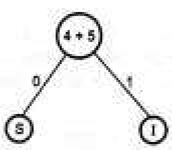


Fig. 11.25 (a)

Next, we replace the two smallest values S(4) and I(5) with their sum, getting a new sequence (7, 8, 9, 12, 29).

O	7
T	8
SI	9
P	12
E	29

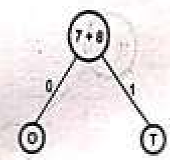


Fig. 11.25 (b)

We again take the two smallest values, O and T, and construct a labelled binary tree. We now have the frequencies (15, 9, 12, 29) which must be sorted into (9, 12, 15, 29).

SI	9
P	12
OT	15
E	29

and the two lowest, which are SI (9), P(12) are selected once again.

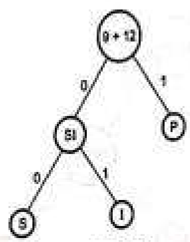


Fig. 11.25 (c)

We now have the frequencies (21, 15, 29) which must be sorted into (15, 21, 29).

OT	15
SIP	21
E	29

Now, we combine the two lowest which are OT (15) and SIP (21).

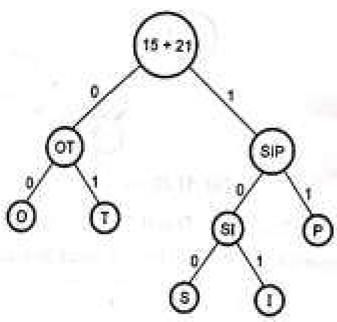


Fig. 11.25 (d)

The two remaining frequencies, 36 and 29 are now combined into the final tree

E	29
OTSIP	36

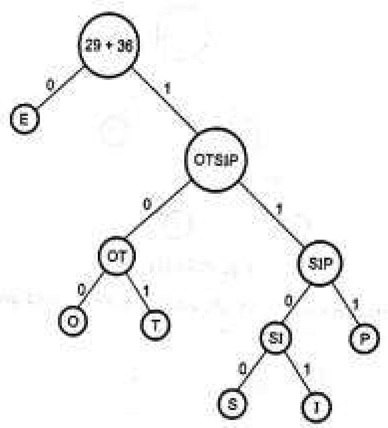


Fig. 11.25 (e)

So looking at the frequency of the letters

S	4	1100
I	5	1101
O	7	100
T	8	101
P	12	111
E	29	0

Highest occurring has less bits.

Example 11.16 *For the following set of weights construct an optimal binary prefix code. For each weight in the set give corresponding code word.*

5, 7, 8, 15, 35, 40.

Solution : Optimal binary prefix code for data 5, 7, 8, 15, 35, 40.

Code for 5 10110

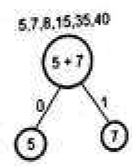


Fig. 11.26 (a)

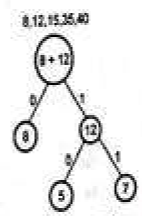


Fig. 11.26 (b)

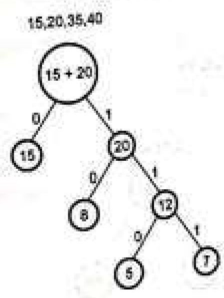


Fig. 11.26 (c)

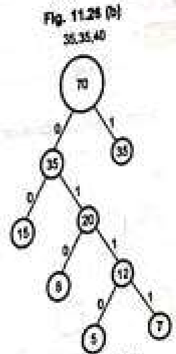


Fig. 11.26 (d)

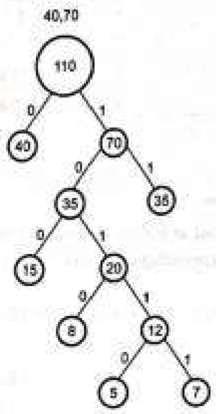


Fig. 11.26 (e)

Code for 7	10111
Code for 8	1010
Code for 15	100
Code for 35	11
Code for 40	0

Example 11.17 *For the following set of weights, construct optimal binary prefix code. For each weight in the set, give the corresponding code words.*
8, 9, 12, 14, 16, 19.

Solution : Optimal binary prefix code for 8, 9, 12, 14, 16, 19.

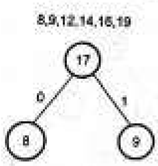


Fig. 11.27 (a)

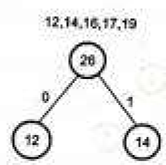


Fig. 11.27 (b)

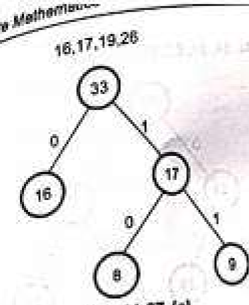


Fig. 11.27 (c)

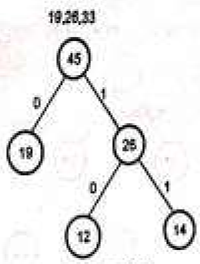


Fig. 11.27 (d)

Code word for 16	00
Code word for 8	010
Code word for 9	011
Code word for 19	10
Code word for 12	110
Code word for 14	111

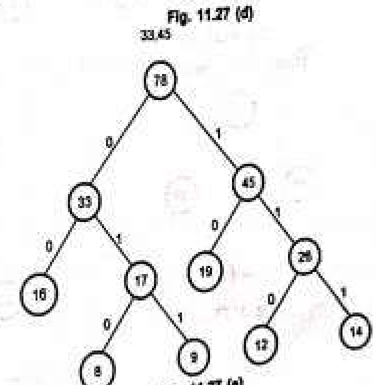


Fig. 11.27 (e)

Example 11.18 *A secondary storage media contains information in files with different formats. The frequency of different types of files is as follows.*
Exe (20), bin (75), bat (20), jpeg (85), dat (51), doc (32), sys (26), c (19), cpp (25), bmp (30), avi (24), prj (29), 1st (35), zip (37).
Construct the Huffman code for this.

Solution :

Fig. 11.28 (a)

37, 39, 44, 51, 51, 59, 67, 75, 85

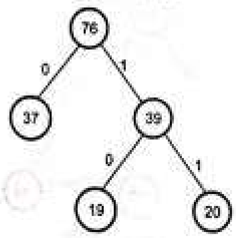


Fig. 11.28 (b)

44, 51, 51, 59, 67, 75, 76, 85

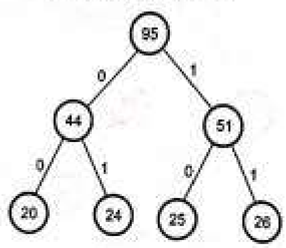


Fig. 11.28 (c)

51, 59, 67, 75, 76, 85, 95

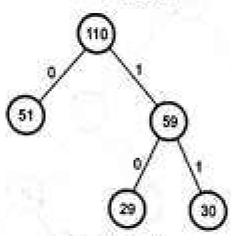


Fig. 11.28 (d)

67, 75, 76, 85, 95, 110

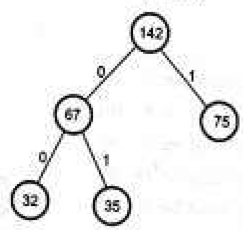


Fig. 11.28 (e)

76, 85, 95, 110, 142

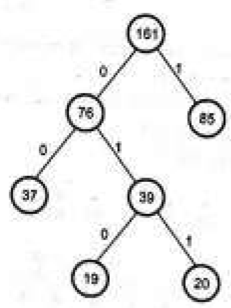


Fig. 11.28 (f)

95, 110, 142, 161

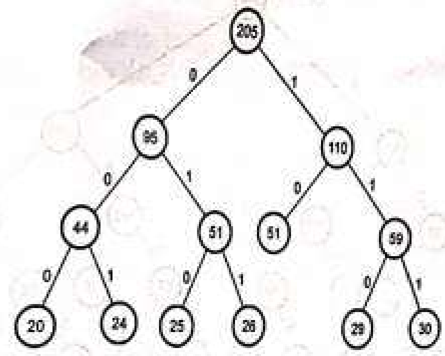


Fig. 11.28 (g)

142, 161, 205

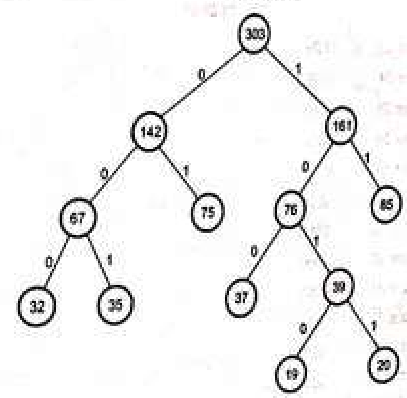


Fig. 11.28 (h)

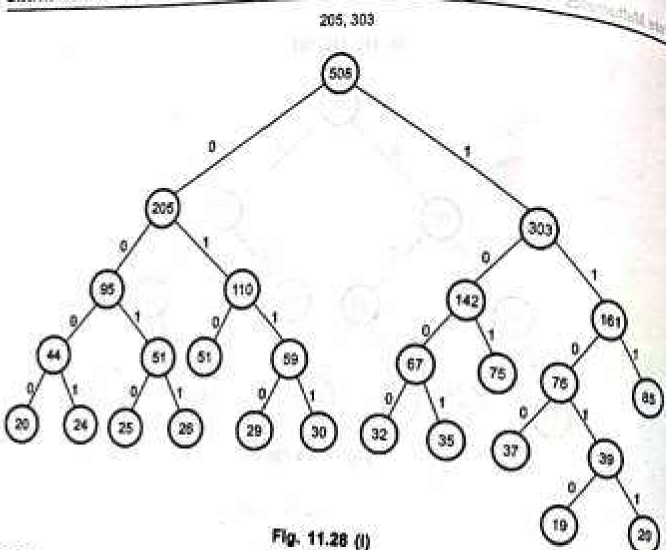


Fig. 11.28 (i)

Code word for 20	0000
Code word for 24	0001
Code word for 25	0010
Code word for 26	0011
Code word for 51	010
Code word for 29	0110
Code word for 30	0111
Code word for 32	1000
Code word for 35	1001
Code word for 75	101
Code word for 37	1100
Code word for 19	11010
Code word for 20	11011
Code word for 85	111

Example 11.19 *For the following sets of weights, construct an optimal binary prefix code. For each weight in the set, give the corresponding code word*

i) 2, 3, 5, 7, 9, 13

ii) 8, 9, 10, 11, 13, 15, 22

Solution : Optimal binary prefix code for data

i) 2, 3, 5, 7, 9, 13

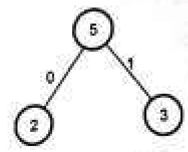


Fig. 11.29 (a)

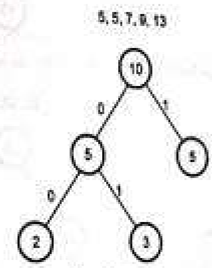


Fig. 11.29 (b)

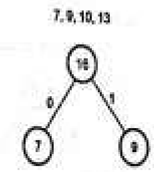


Fig. 11.29 (c)

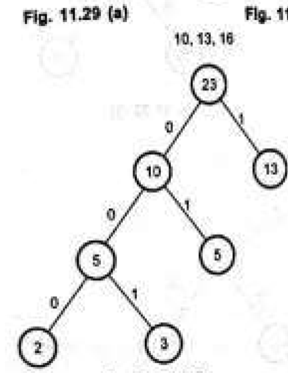


Fig. 11.29 (d)

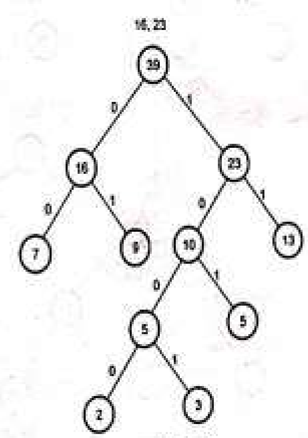


Fig. 11.29 (e)

Code word for 7	00
Code word for 9	01
Code word for 2	1000
Code word for 3	1001
Code word for 5	101
Code word for 13	11

ii) 8, 9, 10, 11, 13, 15, 22

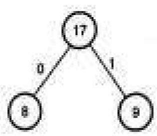


Fig. 11.29 (f)

10, 11, 13, 15, 17, 22

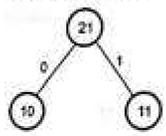


Fig. 11.29 (g)

13, 15, 17, 21, 22

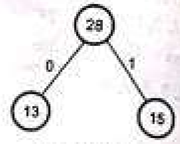


Fig. 11.29 (h)

17, 21, 22, 28

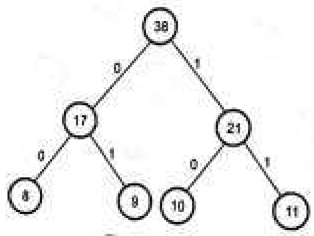


Fig. 11.29 (i)

22, 28, 38

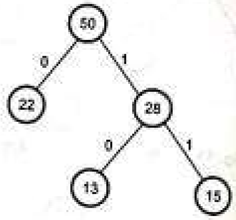


Fig. 11.29 (j)

38, 50

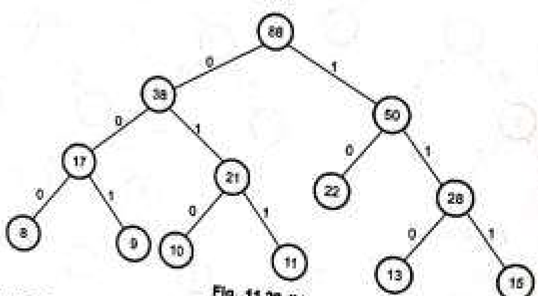


Fig. 11.29 (k)

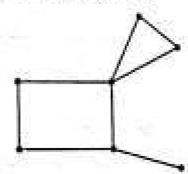
Code word for 8	000
Code word for 9	001
Code word for 10	010

Code word for 11	011
Code word for 22	10
Code word for 13	110
Code word for 15	111

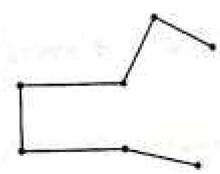
11.8 Spanning Tree

A spanning tree T of a connected, undirected graph G is a tree composed of all the vertices and some (or perhaps all) of the edges of G. Informally, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are formed.

Example



(a) Graph G



(b) T_1

(c) T_2

Fig. 11.30 Spanning trees of graph G

11.8.1 Minimum Spanning Tree

Given a connected, undirected graph, a spanning tree of that graph is a subgraph which is a tree and connects all the vertices together. A single graph can have many different spanning trees. We can also assign a weight to each edge, and use this to assign a weight to a spanning tree by computing the sum of the weights of the edges in that spanning tree. A **minimum spanning tree** or **minimum weight spanning tree** is then a spanning tree with weight less than or equal to the weight of every other spanning tree. More generally, any undirected graph has a minimum spanning forest.

One example would be cable TV company laying cable to a new neighbourhood. If it is constrained to bury the cable only along certain paths, then there would be a graph representing which points are connected by those paths. Some of those paths might be more expansive, because they are longer, or require the cable to be buried deeper these paths would be represented by edges with larger weights. A spanning tree for that graph would be a subset of those paths that has no cycles but still connects to every house.

There might be several spanning trees possible.

A minimum spanning tree would be one with the lowest total cost.

In case, of a tie, there could be several minimum spanning trees. However, one theorem states that if each edge has a distinct weight, the minimum spanning tree is unique. This is true in many realistic situations, such as the one above, where it's unlikely any two paths have exactly the same cost.

11.8.1.1 Prim's Algorithm

Let G = (V, E) be a connected weighted graph.

We construct a minimal spanning tree T for G inductively as follows :

Step 1 : Take a vertex v_0 in graph G.

Set $T = \{\{v_0\}, \varnothing\}$

Step 2 : Find edge $e_1 = (v_0, v_1)$ in E such that its one end vertex v_0 is in set T and its weight is minimum.

Now Set $T = \{\{v_0, v_1\}, \{e\}\}$

Step 3 : Choose next edge $e_k = (v_k, v_l)$ such that its one end vertex v_k is in the set T and other end vertex v_l should not be in T.

Add vertex v_l to the vertex set of T and edge e_k to edge set of T.

Step 4 : Repeat step 3 until T contains all the vertices of G.

The set T now gives minimum spanning tree of graph G.

11.8.1.2 Kruskal's Algorithm

Let G = (V, E) be a weighted connected graph.

Step 1 : Pick up an edge e_i of G such that its weight $w(e_i)$ is minimum.

Step 2 : If edges $e_1, e_2, \ldots, e_k$ have been chosen, then pick an edge e_{k+1} such that

i) $e_{k+1} \neq e_i$ for any $i = 1, 2, \ldots, k$.

ii) The edges $e_1, e_2, \ldots, e_k, e_{k+1}$ do not form a circuit.

iii) The weight $w(e_{k+1})$ is as small as possible subject to condition (ii).

Step 3 : Stop, when (step 2) cannot be implemented.

Remark

This algorithm first appeared in proceedings of the American Mathematical Society in 1956 and was written by Joseph Kruskal.

Example 11.20 *Give the stepwise construction of minimum spanning tree for the following graph using Prim's algorithm.*

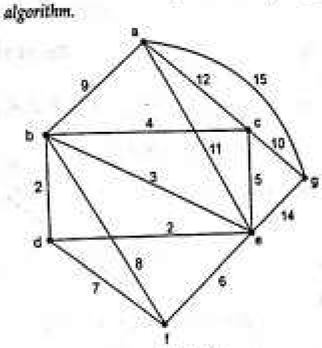


Fig. 11.31

Solution : The construction of minimum spanning tree, using Prim's algorithm. Starting from vertex a,

Step 1

$T = \{\{a\}, \varnothing\}$

a •

Step 2

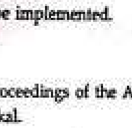


Fig. 11.31 (a)

Now $T = \{\{a, b\}, e_1\}$

Step 3

Fig. 11.31 (b)

Now $T = \{\{a, b, d\}, \{e_1, e_2\}\}$

Step 4

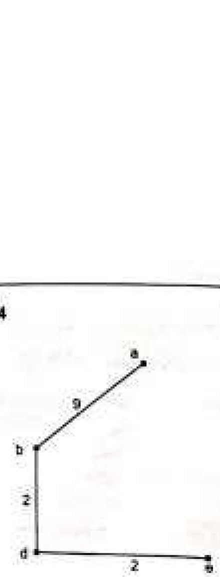


Fig. 11.31 (c)

Now $T = \{\{a, b, d, e\}, \{e_1, e_2, e_3\}\}$

Step 5

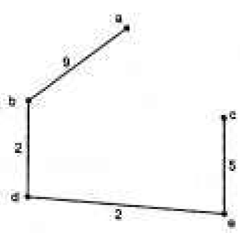


Fig. 11.31 (d)

Now $T = \{\{a, b, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$

Step 6

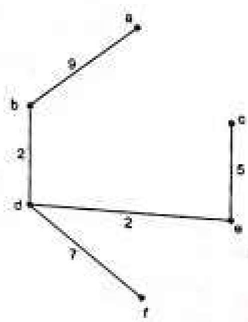


Fig. 11.31 (e)

Now $T = \{\{a, b, d, e, c, f\}, \{e_1, e_2, e_3, e_4, e_5\}\}$

Step 7

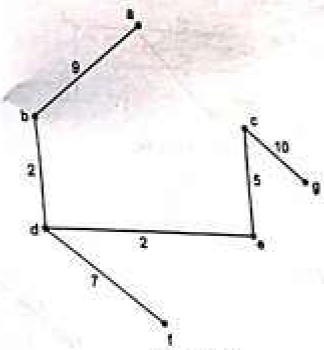


Fig. 11.31 (f)

The graph obtained is the minimum spanning tree of weight 35.

Example 11.21 *Determine minimum spanning tree for the given graph using Prims algorithm.*

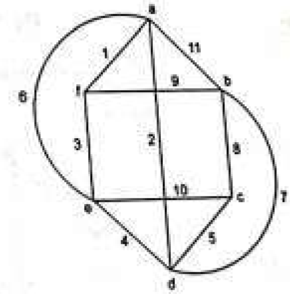


Fig. 11.32

Solution : The construction of minimum spanning tree using Prim's algorithm.

Starting from vertex a.

Step 1

$$T = \{\{a\}, \varnothing\}$$

a •

Step 2

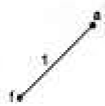


Fig. 11.32 (a)

$T = \{\{a, f\}, e_1\}$

Step 3

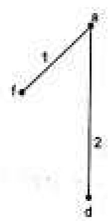


Fig. 11.32 (b)

$T = \{\{a, f, d\}, \{e_1, e_2\}\}$

Step 4

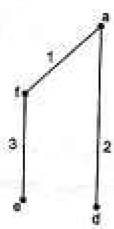


Fig. 11.32 (c)

$T = \{\{a, f, d, e\}, \{e_1, e_2, e_3\}\}$

Step 5

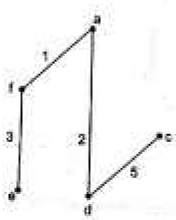


Fig. 11.32 (d)

$T = \{\{a, f, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$

Step 6

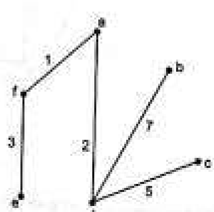


Fig. 11.32 (e)

$T = \{\{a, f, d, e, c, b\}, \{e_1, e_2, e_3, e_4, e_5\}\}$

The graph obtained is the minimum spanning tree of weight 18.

Example 11.22 *Obtain minimum spanning tree using Prim's algorithm.*

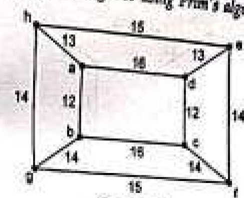


Fig. 11.33

Solution : Step 1

Fig. 11.33 (a)

Step 2

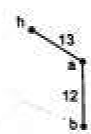


Fig. 11.33 (b)

Step 3

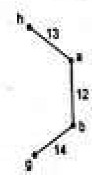


Fig. 11.33 (c)

Step 4

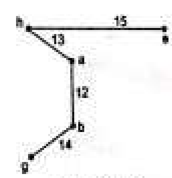


Fig. 11.33 (d)

Step 5

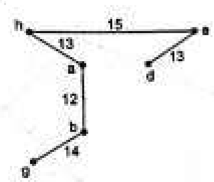


Fig. 11.33 (e)

Step 6

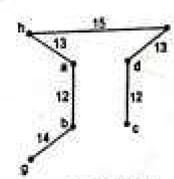


Fig. 11.33 (f)

Step 7

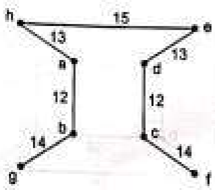


Fig. 11.33 (g)

The graph obtained is the minimum spanning tree of weight 93.

Example 11.23 *Use the Kruskal's algorithm to find the minimum spanning tree for the graph shown.*

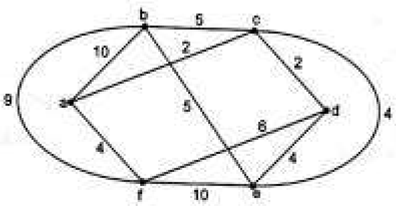


Fig. 11.34

Solution :

Step 1

Fig. 11.34 (a)

Step 2

Fig. 11.34 (b)

Step 3

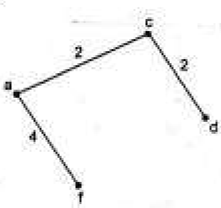


Fig. 11.34 (c)

Step 4

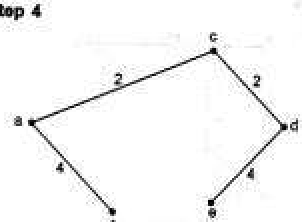


Fig. 11.34 (d)

Step 5

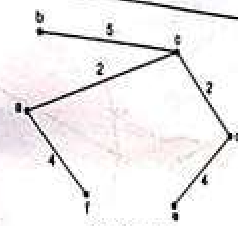


Fig. 11.34 (e)

The graph is a minimum spanning tree and its total weight is 17.

Example 11.24 *Obtain the minimum spanning tree for the following graph. Obtain the total cost of minimum spanning tree.*

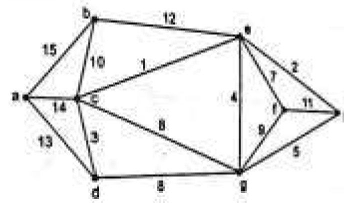


Fig. 11.35

Solution : Using Kruskal algorithm, the minimum spanning tree is obtained as follows :

1.

Fig. 11.35 (a)

2.

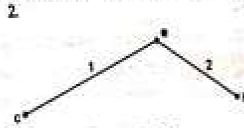


Fig. 11.35 (b)

3.

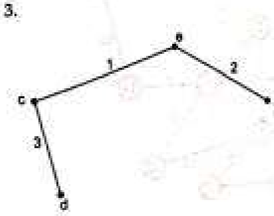


Fig. 11.35 (c)

4.

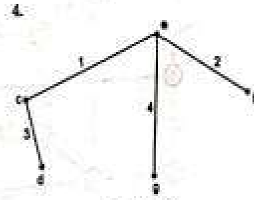


Fig. 11.35 (d)

5.

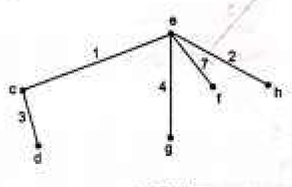


Fig. 11.35 (e)

6.

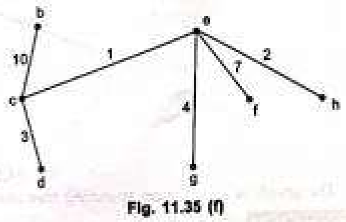


Fig. 11.35 (f)

7.

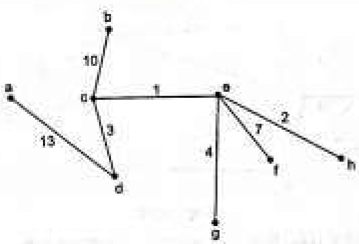


Fig. 11.35 (g)

The graph in (7) is a minimum spanning tree of the given graph. Its total cost is 40.

Example 11.25 *Find the minimum spanning tree using*
i) Kruskal's algorithm ii) Prim's algorithm.

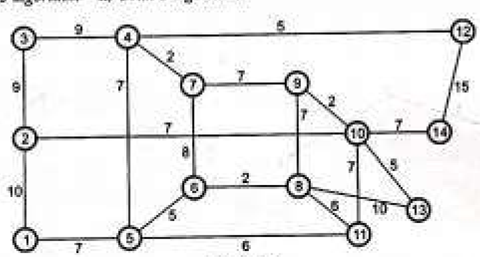


Fig. 11.36

Solution :

i) Kruskal's algorithm

Step 1

Fig. 11.36 (a)

Step 2

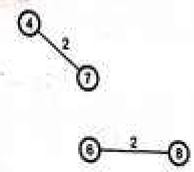


Fig. 11.36 (b)

Step 3

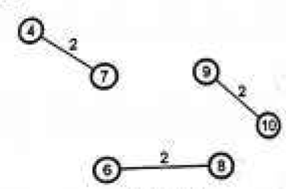


Fig. 11.36 (c)

Step 4

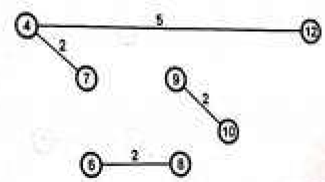


Fig. 11.36 (d)

Step 5

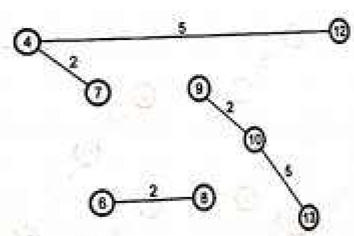


Fig. 11.36 (e)

Step 6

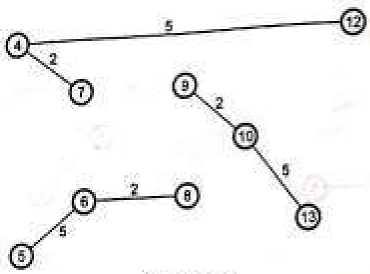


Fig. 11.36 (f)

Step 7

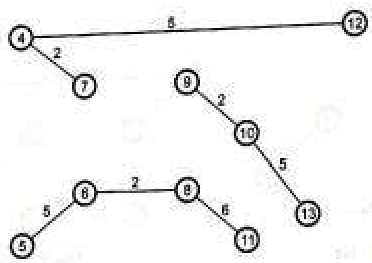


Fig. 11.36 (g)

Step 8

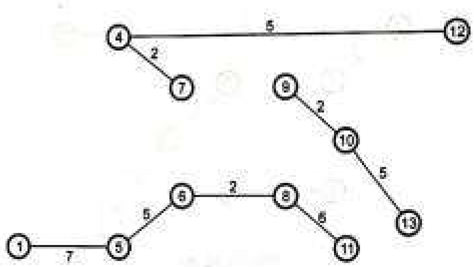


Fig. 11.36 (h)

Step 9

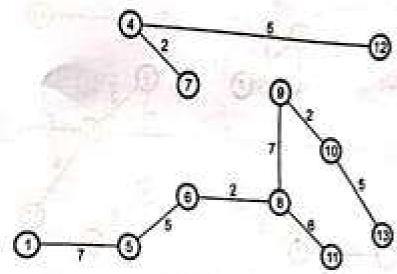


Fig. 11.36 (i)

Step 10

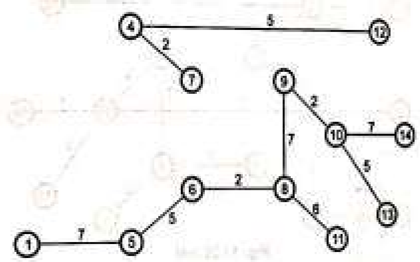


Fig. 11.36 (j)

Step 11

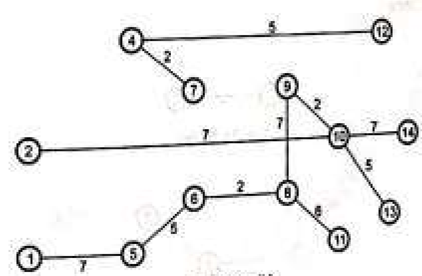


Fig. 11.36 (k)

Step 12

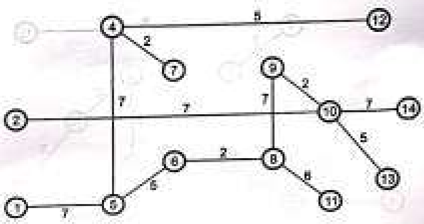


Fig. 11.36 (l)

Step 13

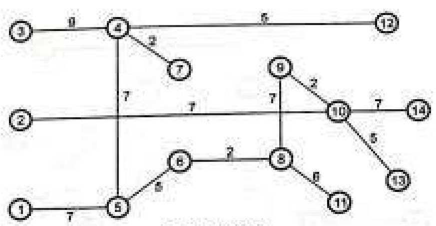


Fig. 11.36 (m)

The graph in Fig. 11.36 (m) is a minimum spanning tree with weight 71.

ii) Prim's algorithm

Step 1

Fig. 11.36 (n)

Step 2

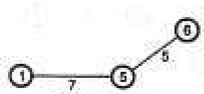


Fig. 11.36 (o)

Step 3

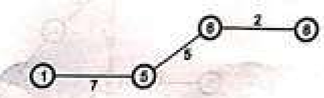


Fig. 11.36 (p)

Step 4

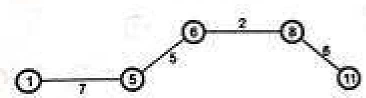


Fig. 11.36 (q)

Step 5

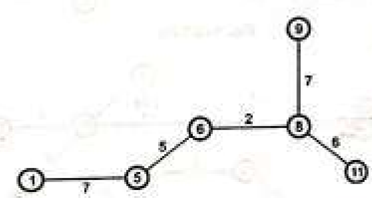


Fig. 11.36 (r)

Step 6

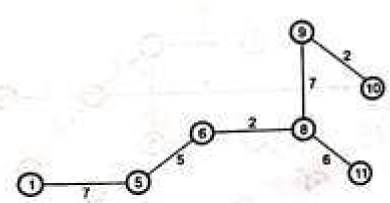


Fig. 11.36 (s)

Step 7

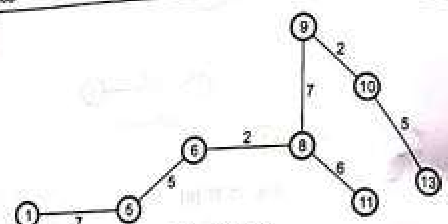


Fig. 11.36 (t)

Step 8

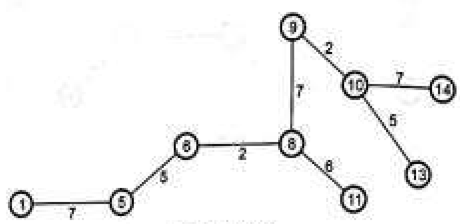


Fig. 11.36 (u)

Step 9

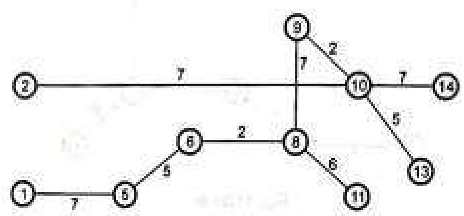


Fig. 11.36 (v)

Step 10

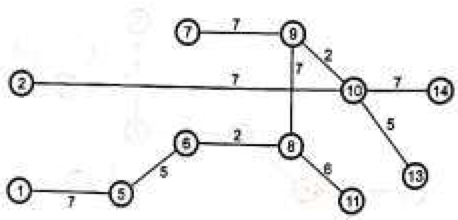


Fig. 11.36 (w)

Step 11

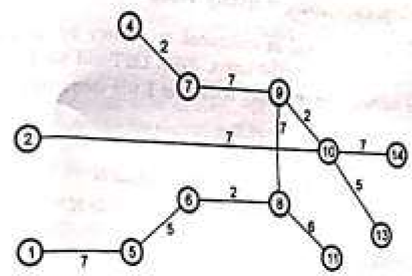


Fig. 11.36 (x)

Step 12

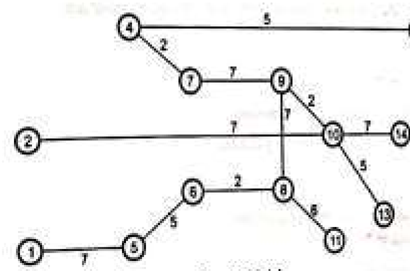


Fig. 11.36 (y)

Step 13

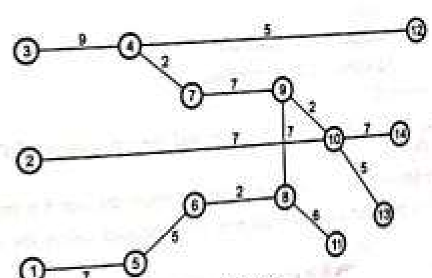


Fig. 11.36 (z)

The graph in Fig. 11.36 (z) is minimum spanning tree of the given graph with weight 71.

11.9 Linked Representation of Binary Trees

Let T be a binary tree. T will be maintained in memory by means of a linked representation which uses three parallel arrays, INFO, LEFT and RIGHT and a pointer variable ROOT as follows. First of all each node N of T will correspond to a location K such that.

1. INFO [K] contains the data at the node N.
2. LEFT [K] contains the location of the left child of node N.
3. RIGHT [K] contains the location of the right child of node N.

11.9.1 Tree Traversal

There are three standard ways of traversing a binary tree T with root R.

These three algorithms called preorder, inorder and post order are

1. Pre Order

1. Process the root R.
2. Traverse the left subtree of R in preorder.
3. Traverse the right subtree of R in preorder.

2. In Order

1. Traverse the left subtree of R in inorder.
2. Process the root R.
3. Traverse the right subtree of R in inorder.

3. Post order

1. Traverse the left subtree of R in postorder.
2. Traverse the right subtree of R in postorder.
3. Process the root R.

Each algorithm contains the same three steps and the left subtree of R is always traverese before the right subtree.

The difference between the algorithms is the time at which the root R is processed.

Specially, in the preorder algorithm, the root R is processed before the subtree are taveresed.

In inorder algorithm the root R is processed between the traversal of subtrees.

In postorder algorithm the root R is processed after the subtrees are traversed.

Application of graph

1. In computer science graphs are used to represent the flow of computation.
2. Google maps uses graphs for building transportation system, where intersection of two or more roads are considered to be a vertex and the road connecting two vertices is considered to be an edge, thus their navigation system is based on the algorithm to calculate the shortest path between two vertices.

Application of List structures

1. Implementation of graphs - Adjacency list representation of graphs is most popular which uses linked list to store adjacent vertices.

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